Q.1 Consider the sinusoidal waveforms:

\[ x_1(t) = 10 \cos(100\pi t - 45^\circ), \quad x_2(t) = 10 \cos(5000\pi t - 45^\circ) \]

a) Give the time periods of \( x_1(t) \), \( x_2(t) \) and \( x_1(t) + x_2(t) \).

b) \( x_1(t) \) and \( x_2(t) \) are delayed by different amounts of time (from the origin). Calculate these delays.

c) \( x_1(t) \) lags \( x_2(t) \) by \( t_0 \) and \( x_2(t) \) leads \( x_1(t) \) also by \( t_0 \). Calculate \( t_0 \).

d) \( x_1(t) \) and \( x_2(t) \) are delayed by the same amount of phase (from the origin) which is \( 45^\circ \) or \( \pi/4 \) radians. Compare this phase difference with \( t_0 \) given in part (c).

e) Considering any two sinusoidal signals, in which case can you state the following?

\[ x_1(t) \text{ lags } x_2(t) \text{ by } \theta_0 \text{ (or } x_2(t) \text{ leads } x_1(t) \text{ by } \theta_0) \text{, where } \theta_0 \text{ is the phase (angle) difference between the two signals} \]

Q.2 Consider the SISO system in Fig.1.

a) Is this system linear? Why?

b) Is this system time invariant. Why?

\[ m(t) \rightarrow s(t) = m(t) 10 \cos(2 \times 10^5 \pi t) \]

Fig.1

Q.3 Let \( m(t) = 8 \cos(5 \times 10^3 \pi t) \).

a) Show that \( s(t) \) is periodic and calculate its time period and frequency.

b) Give an approximate sketch of \( s(t) \).

Q.4 Consider the sinusoidal waveforms:

\[ x_1(t) = 10 \cos(100\pi t), \quad x_2(t) = 20 \cos(50\pi t - \frac{\pi}{6}), \quad x_3(t) = 15 \sin(15\pi t) \]

a) Give the time periods of \( x_1(t) \), \( x_2(t) \), \( x_3(t) \) and \( x(t) = x_1(t) + x_2(t) + x_3(t) \).
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b) Consider the Fourier series expansion of \( x(t) \) and give \( a_k, \phi_k, c_k, a_k \) and \( b_k \).

Q.5 Consider the decaying exponential waveform

\[ e(t) = Ae^{-\alpha t}u(t); \alpha > 0 \]

where \( t = \tau = \frac{1}{\alpha} \) is called the time constant of \( e(t) \).

a) Show that at time \( t = N\tau \)

\[ x(N\tau) = \frac{x(0)}{e^N}, \text{ i.e., } \frac{x(N\tau)}{x(0)} = \frac{1}{e^N} \]

b) Now consider the sinusoidal waveform with exponential envelope:

\[ x(t) = \{Ae^{-\alpha t} \cos(\omega_0 t + \varphi)\} u(t); \alpha > 0 \]

Give the relationship between the time period \( T_0 \) of the sinusoidal signal and the time constant \( \tau \) of the exponential signal such that

\[ \frac{x(10T_0)}{x(0)} = 0.01 \]

Q.6 (a) The impulse response of an LTI system is given as

\[ h(t) = \beta e^{-\alpha t} \cos(\omega_0 t + \varphi)u(t); \alpha > 0 \]

Consider \( h(t) \) to be the product of the two signals \( f(t) \) and \( g(t) \) such that

\[ h(t) = f(t)g(t) \]

where \( f(t) = e^{-\alpha t}u(t) \) and \( g(t) = \beta \cos(\omega_0 t + \varphi) = \beta \cos(\omega_0 (t + \frac{\varphi}{\omega_o})) \)

Using the frequency convolution property, or

\[ F(j\omega) * \delta(\omega \pm \omega_0) = F(\omega \pm \omega_0) \]

show that

\[ H(j\omega) = \beta \frac{(\alpha \cos \varphi - \omega_o \sin \varphi) + j(\cos \varphi)\omega}{(\alpha + j\omega)^2 + \omega_o^2} \]

\[ F(j\omega) = \frac{1}{\alpha + j\omega} \quad \text{and} \quad F(\cos \omega_0 t) = \pi \delta(\omega + \omega_o) + \pi \delta(\omega - \omega_o) \]
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Hint: To obtain the same result use also the time shifting property for
\[ \cos(\omega_0 t + \varphi) = \cos \omega_0 (t + \frac{\varphi}{\omega_0}). \]

(b) If
\[ H(j\omega) = \frac{j\omega}{1 - \omega_0^2 + j\omega}\]
evaluate \(\alpha\), \(\beta\), \(\varphi\) and \(\omega_o\) and express \(h(t)\) using these values.

(e) Let the input to the an LTI system with the transfer function given in part (b) be of the form
\[ x(t) = \frac{1}{2} + \frac{4}{\pi^2} \left( \cos t + \frac{1}{9} \cos 3t + \frac{1}{25} \cos 5t + \ldots \right) \]
Evaluate the first four terms of the Fourier series of the output \(y(t)\).

Q.7 Consider the finite duration signal \(f(t)\) given in Fig.2.

(a) Evaluate the Fourier transform \(F(j\omega)\) of \(f(t)\) and show that
\[ F(j\omega) = \tau \frac{\sin \frac{\omega \tau}{2}}{\omega \tau} \]
Using the relation between \(F(j\omega)\) and \(c_k\), evaluate \(c_k\) for the periodic signal given in Fig.3.

(b) If \(\frac{T_0}{\tau} = N\), where \(N\) is an integer, show that \(c_k = 0\) for \(k = mN\) for \(m = \pm 1, \pm 2, \pm 3\ldots\). For \(N = 3\) plot \(c_k\). Using the relation between \(\alpha_k\) and \(c_k\), i.e. \(\alpha_0 = c_0\), \(\alpha_k = 2|c_k|\), \(\varphi_k = \arg\{c_k\}\), express \(f_p(t)\) as a sum of real sinusoidal signals and write the sum explicitly up to and including the 5th term.
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(b) The periodic signal given in Fig. 4 is related to $f_p(t)$ as

$$g_p(t) = 2f_p(t - \frac{\tau}{2}) - 1$$

![Fig. 4](image)

Using $c_k$ obtained in part (b) and the shifting property of Fourier transforms find the Fourier series coefficients $c_k$ of $f_p(t - \tau)$. Using the result obtained in part (b), the $c_k$ of $f_p(t - \tau)$ and the relation between $f_p(t)$ and $g_p(t)$, express $f_p(t)$ as a sum of real sinusoidal functions and write the sum explicitly up to and including the 5th term.

Q. 8 (a) A band-pass filter is given by the integro-differential equation

$$v_1(t) - v_2(t) = \frac{dv_2(t)}{dt} + \int_0^t v_2(t)dt$$

Evaluate

$$H(j\omega) = \frac{V_j(j\omega)}{V_i(j\omega)}$$

and show that

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega - \frac{1}{\omega})^2}}$$

Plot $|H(j\omega)|$ versus $\omega$.

(b) Given

$$F\{e^{-\alpha t} \cos(\omega t + \varphi)u(t)\} = \cos \varphi \frac{\alpha - \omega \tan \varphi + j\omega}{(\alpha + j\omega)^2 + \omega^2}$$

Compare this with $|H(j\omega)|$ obtained in part (a) and determine the numerical values of $\alpha$, $\Omega_0$ and $\varphi$. Give the impulse response of the band-pass filter as a function of time.
Q.9  Consider the rectangular pulse waveform \( p(t) \) in Fig.5a whose Fourier transform is given by

\[
P(j\omega) = \frac{\sin \frac{a\omega}{2}}{\omega}.
\]

(a) Using \( P(j\omega) \) and the relationship between the pulse \( p(t) \) and the doublet pulse \( x(t) \) given in Fig.5b, evaluate the Fourier transform of \( x(t) \).

(b) Consider the finite duration triangular pulse waveform \( y(t) \) given in Fig.5c and the relationship between \( x(t) \) and \( y(t) \). \( Y(j\omega) \) is sketched in Fig.7. Use the integration-in-time property of Fourier transform to show that the Fourier transform of \( y(t) \) is given by

\[
Y(j\omega) = a^2 \left( \frac{\sin \frac{a\omega}{2}}{a\omega} \right)^2.
\]

Q.10  Consider the periodic signal given in Fig.6.

(a) Using the Fourier transform \( Y(j\omega) \) given above evaluate, as a function of \( k \), the Fourier series coefficients \( y_k \) of the periodic signal \( y_p(t) \) given in Fig.6, where \( k \) is any integer representing the harmonics of \( y_p(t) \).

(b) Using the integration-in-time-property between \( X(j\omega) \) and \( Y(j\omega) \), obtain the relationship between \( c_k^y \) and \( c_k^x \), where \( c_k^x \) the Fourier series coefficients of the periodic signal given in Fig.8. Take \( a=1 \) and sketch \( c_k^y \) and \( \text{Im}\{c_k^y\} \) versus \( k \).
Solutions for
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(c) Using \( c_k^y \) and \( c_k^x \) express \( y_p(t) \) and \( x_p(t) \) as sums of real sinusoidal components and verify that \( \frac{dy_p(t)}{dt} = x_p(t) \)

![Figure 6](image)

\[ y_p(t) = a \]

\[ -a \]

\[ 0 \]

\[ a \]

\[ t \]

![Figure 7](image)

\[ Y(j\Omega) = a^2 \left( \frac{\sin \frac{a\Omega}{2}}{\frac{a\Omega}{2}} \right)^2 \]

\[ 0 \]

\[ 2\pi/a \]

\[ 4\pi/a \]

\[ 6\pi/a \]

\[ \Omega \]

![Figure 8](image)

Q.11 The Fourier transform of the triangular waveform given in Fig.9.a is given as:

\[ F(j\omega) = \left( \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^2 \]

(a) Using \( F(j\omega) \) evaluate the Fourier transforms of the waveforms in Fig.9.b-d.
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(b) Give the Fourier transform of \( f(t) \cos \omega_0 t \) where \( f(t) \) is the waveform given in Fig.9.a.

Q.12 Consider the discrete-time LTI system given by the difference equation

\[ y(n) = 1.08y(n-1) + K\delta(n) - x(n) \]

(a) Sketch a block diagram realising the difference equation given above.

(b) Let

\[ x(n) = \begin{cases} r & \text{for } 1 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases} \]

and evaluate \( y(n) \) in terms of \( K \) and \( r \) where the initial condition is given as \( y(-1)=0 \).

(c) Let \( K = 10 \) and show that \( y(n) = 0 \) for \( n \geq 4 \) if \( r = 3.019 \). Evaluate and plot \( y(n) \) for \( n = 0,1,2,3,4,5,6,7 \ldots \)

(d) Is this system stable?

Q.13 Consider the discrete-time LTI system given by the difference equation

\[ y(n) - \frac{\sqrt{3}}{2} y(n-1) + \frac{1}{4} y(n-2) = x(n) - x(n-2) \]

and the arbitrary initial conditions

\[ y(-1) = 1 \]
\[ y(-2) = -1 \]

(a) Evaluate the impulse response and the unit-step response for \( n = 0,1,2,\ldots,7 \) (take \( y(-1)=y(-2)=0 \)).
b) Evaluate the zero-input response for the initial conditions given above for n=0,1,2,...,7.

c) For the unit-step input determine zero-state response and the complete response.

d) Sketch the simulation of the first canonical form of the difference equation given above.

e) For the above canonical form obtain the state equations.

f) Repeat part (a-c) using the state equations.

Q.14 Consider the low-pass filter with the impulse response:

\[ h(t) = e^{-\alpha t}u(t) \]

(a) Using the Fourier transform of \( h(t) \), show that the frequency response of the low-pass filter is given by:

\[ H(j\omega) = \frac{1}{\alpha + j\omega} \]

Plot \(|H(j\omega)|\) and show that \( \alpha \) is the 3db cutoff frequency \( \omega_c \).

(b) Show using the Modulation Property of Fourier transform that:

\[ h_{bp}(t) = h(t)\cos{\omega_o t} \]

is the impulse response of a bandpass filter with the centre frequency \( \omega_o \), provided that:

\[ \frac{\omega_o}{\omega_c} = \frac{F_c}{F_o} << 1 \]

Plot \(|H_{bp}(j\omega)|\).

Q.15 The square wave \( p_r(t) \) in Fig.10 is used for Pulse Amplitude Modulation (PAM).

The Fourier series expansion for \( p_r(t) \) is given as:

\[ p_r(t) = \frac{1}{2} + \frac{2}{\pi} \cos(\omega_o t) - \frac{2}{3\pi} \cos(3\omega_o t) + \frac{2}{5\pi} \cos(5\omega_o t) - \frac{2}{7\pi} \cos(7\omega_o t) + \ldots \]
Solutions for
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(a) Let the message signal be represented by \( f(t) \). Using the above expression for \( p, t(t) \) write down an expression for the PAM signal \( p, t(t)f(t) \). Using the modulation property of the Fourier transform give the Fourier transform of the PAM signal.

(b) Sketch the frequency spectrums of \( f(t) \), \( p, t(t) \) and \( p, t(t)f(t) \).

(c) Show how the Double Side-Band Suppressed Carrier (DSB-SC) modulated signal can be generated from the PAM signal.

Q.16 The Fourier series expansion of the periodic signal \( v(t) \) is given as:

\[
v(t) = \alpha_0 + \sum_{k=1}^{\infty} \alpha_k \cos(k\omega_f t - \phi_k)
\]

(a) Assuming that \( v(t) \) is the voltage across a 1ohm resistor, show that the average power developed during one period is given by:

\[
P = \alpha_0^2 + \sum_{k=1}^{\infty} \frac{1}{2} \alpha_k^2
\]

Using \( P \) and the following definition write down the expression giving the rms value of \( v(t) \), \( V_{rms} \):

\[
V_{rms} = \sqrt{\frac{1}{T_f} \int_0^{T_f} v^2(t) dt}
\]

where \( T_f \) is the fundamental period of \( v(t) \).

(b) Let \( v(t) \) be the periodic triangular wave in Fig.6 whose Fourier series expansion is given as:

\[
v(t) = \frac{1}{2} + \frac{4}{\pi^2} \left( \cos(\omega_1 t) + \frac{1}{9} \cos(3\omega_1 t) + \frac{1}{25} \cos(5\omega_1 t) + \ldots \right)
\]

Compute the average power of \( v(t) \):

(i) Directly from Fig.6;

(ii) Using the expression given in part (a) by truncating the series after the first five terms given above.

(iii) Compare the values obtained in (i) and (ii) and explain whether the truncated series is a sufficiently accurate representation of \( v(t) \).
Q.17 Consider the square wave in Fig.11 whose truncated Fourier series expansion is given as:

\[ v_p(t) = a_0 + a_1 \cos(\omega_0 t) + a_3 \cos(3\omega_0 t) + a_5 \cos(5\omega_0 t) + a_7 \cos(7\omega_0 t) \]

where

\[
a_0 = \frac{1}{2}, \quad a_1 = \frac{2}{\pi}, \quad a_3 = -\frac{2}{3\pi}, \quad a_5 = \frac{2}{5\pi}, \quad a_7 = -\frac{2}{7\pi}
\]

(a) Assuming that \( v_p(t) \) is the voltage across a 1ohm resistor and knowing that the average power \( P \) delivered to this resistor is given by:

\[
P = \frac{1}{T_0} \int_0^{T_0} v_p^2(t) dt
\]

prove that the rms value of \( v(t) \) is given by

\[
V_{rms} = \sqrt{\frac{1}{T_0} \int_0^{T_0} v_p^2(t) dt}
\]

(b) Compute the average power \( P \):

(i) Directly from Fig.11.
(ii) Using the truncated Fourier series expansion given in part(a).
(iii) Compare the values obtained in (i) and (ii) and explain whether the truncated series is a sufficiently accurate representation of \( v_p(t) \).

Q.18 Consider the finite-duration signal \( v(t) \) in Fig.12.
(c) Evaluate $V(j\omega)$ and show that

$$V(j\omega) = \frac{T_0}{2} \sin \frac{T_0\omega}{4}$$

(d) Show that the relation between the complex Fourier coefficients $c_k$ of $v_p(t)$ given in Q.17 and the Fourier transform $V(j\omega)$ of $v(t)$ is given as:

$$c_k = \frac{1}{T_0} V(jk\omega_0)$$

Using $V(j\omega)$ and the relationship given above evaluate $c_k$.

(e) Also evaluate $\alpha_k, \phi_k, a_k$ and $b_k$ and present them in the following table and confirm with the coefficients given in Q.17.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$c_k$</th>
<th>$\alpha_k$</th>
<th>$\phi_k$</th>
<th>$a_k$</th>
<th>$b_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
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<td>2</td>
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<td>6</td>
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<tr>
<td>7</td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

Q.19 A filter transfer function is given by:

$$H(s) = \frac{\omega_0 s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

(a) Show that the magnitude frequency response is given by:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + Q^2 \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2}}$$

and $H(s)$ represents a band-pass filter with center frequency $\omega_0$. 
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(b) Evaluate the 3-dB bandwith the 3-dB relative bandwith, the lower and upper cutoff frequencies, \( \omega_l \) and \( \omega_u \).

(c) Let \( H(s) = \frac{10^8}{s^2 + 10^5 s + 10^8} \) and the input \( x(t) = \cos 10^3 t + \cos 10^4 t + \cos 10^5 t \).

Find the approximate output \( y(t) \).

Q.20 A bandpass filter given by

\[
H(s) = \frac{\omega_0 s}{Q} \frac{1}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}
\]

is to be designed to pass only the fundamental of the periodic waveform given in Fig.1.

(a) Show that the phase response of the filter is given by:

\[
\phi(\omega) = \begin{cases} 
\frac{\pi}{2} - \arctan \frac{\omega_0}{Q} - \arctan \frac{\omega}{\omega_0} & \omega \geq 0 \\
-\frac{\pi}{2} + \arctan \frac{\omega_0}{Q} - \arctan \frac{\omega}{\omega_0} & \omega \leq 0
\end{cases}
\]

Roughly plot \( \phi(\omega) \) versus \( \omega \).

(b) Determine the value of the quality factor \( Q \) such that the second harmonic of \( v(t) \) remains outside the 3-dB pass-band.

(c) The magnitude response of the filter is given by:

\[
|H(j\omega)| = \frac{1}{\sqrt{1 + Q^2 \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}}
\]

The first four components of the amplitude spectrum and the phase spectrum of the input \( v(t) \) are given as:

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_k )</td>
<td>0.25</td>
<td>0.45</td>
<td>0.32</td>
<td>0.15</td>
</tr>
<tr>
<td>( \phi_k )</td>
<td>-</td>
<td>-\pi/4</td>
<td>-\pi/2</td>
<td>-3\pi/4</td>
</tr>
</tbody>
</table>

Let \( Q = 10 \). Make the same table for the output of the filter and give its time-domain expression.
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Q.21
(a) Given any discrete-time signal \( y(n) \) prove that:
\[
Y^+(z) = Z^+ \{y(n)\} = Z \{y(n)u(n)\}
\]
where \( Z^+ \) and \( Z \) represent the one-sided and double-sided z-transforms, respectively.

(b) Using the one-sided z-transform find the unit-step response of the LTI discrete-time system given by:
\[
y(n) = \alpha y(n-1) + x(n)
\]
and \( y(-1)=1 \).

(c) Identify the zero-input response (ZIR), the zero-state response (ZSR), the natural response (NR), the forced response (FR) and complete the table below:

<table>
<thead>
<tr>
<th>Type of response</th>
<th>Expression</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero-input response</td>
<td>( X(z) )</td>
<td>( X(z) )</td>
</tr>
<tr>
<td>Zero-state response</td>
<td>( y(n) )</td>
<td>( y(n) )</td>
</tr>
<tr>
<td>Natural response</td>
<td>( e(n) )</td>
<td>( e(n) )</td>
</tr>
<tr>
<td>Forced response</td>
<td>( x(n) )</td>
<td>( x(n) )</td>
</tr>
</tbody>
</table>

(d) Plot ZIR, ZSR, NR, FR and the complete response versus \( n \).
\[
Z^+ \{x(n)\} = z^{-k} \left[ X^+(z) + \sum_{n=1}^{k} x(-n)z^n \right],
Z^+ \{u(n)\} = \frac{1}{1-z^{-1}}, Z^+ \{\alpha^n u(n)\} = \frac{1}{1-\alpha z^{-1}}
\]

(e) Repeat (b)-(d) for \( x(n) = u(n-2) \).

Q.22 A discrete-time dynamical LTI system is given by the following input-output difference equation:
\[
y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 e(n) + b_1 e(n-1) + b_2 e(n-2)
\]
where \( y(n) \) and \( e(n) \) represent the output and the input signals, respectively.

(a) Rewrite the above equation in the form
\[
y(n) = -a_2 y(n-2) + b_2 e(n-2) - a_1 y(n-1) + b_1 e(n-1) + b_0 e(n)
\]
and draw the corresponding canonical form simulation by showing the signals at all points on the simulation.

(b) Take \( a_1 = -1, a_2 = 0.5, b_0 = 1, b_1 = 0, b_2 = 1 \) in the difference equation in part (a) and compute the impulse response \( h(n) \) for \( n=0,1,2,\ldots,7 \) (\( y(-1) = 0, y(-2) = 0 \)).

(c) Assign state variables to appropriate internal signals and obtain the corresponding Canonical State Model Equations (or the Canonical State Representation or the Canonical State Realization) corresponding to the difference equation in part (a).

(d) Compute the impulse response using the state equations found in part (b).
SOLUTIONS

Q.6

(a) \[ F \{ f(t) \cdot g(t) \} = \frac{1}{2\pi} F(j\omega) \ast G(j\omega) \]
\[ F(j\omega) = \frac{1}{\alpha + j\omega} \]
\[ G(j\omega) = \beta_\pi \left[ \delta(\omega - \omega_o) + \delta(\omega + \omega_o) \right] e^{j\omega_o \varphi} = \beta_\pi \left[ \delta(\omega - \omega_o) e^{j\varphi} + \delta(\omega + \omega_o) e^{-j\varphi} \right] e^{j\omega_o \varphi} \]

We know that:
\[ F(j\omega) \ast \delta(\omega \pm \omega_o) = F\left( j[\omega \pm \omega_o] \right) \]

Therefore:
\[ H(j\omega) = F \{ f(t) \cdot g(t) \} = \frac{1}{2\pi} \frac{1}{\alpha + j\omega} \ast \left\{ \beta_\pi \left[ \delta(\omega - \omega_o) e^{j\varphi} + \delta(\omega + \omega_o) e^{-j\varphi} \right] \right\} \]
\[ H(j\omega) = \beta \frac{\alpha \cos \varphi - \omega_o \sin \varphi + j\cos(\varphi \omega)}{(\alpha + j\omega)^2 + \omega_o^2} \]

(b) Let
\[ H(j\omega) = \frac{j\omega}{1 - \omega^2 + j\omega} \]

We also have
\[ H(j\omega) = \beta \frac{\alpha \cos \varphi - \omega_o \sin \varphi + j\cos(\varphi \omega)}{\alpha^2 + \omega_o^2 - \omega^2 + j2\alpha \omega} \]

Comparison of the two yields:
\[ \alpha \cos \delta - \omega_o \sin \varphi = 0 \quad \Rightarrow \quad \tan \varphi = \frac{\alpha}{\omega_o} \]
\[ \beta \cos \varphi = 1 \quad \Rightarrow \quad \cos \varphi = \frac{1}{\beta} \]
\[ \alpha^2 + \omega_o^2 = 1 \]
\[ 2\alpha = 1 \quad \Rightarrow \quad \alpha = \frac{1}{2} \quad \omega_o = \frac{\sqrt{3}}{2} \]

Therefore,
\[ \tan \varphi = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \varphi = 30^\circ \]
\[ \cos \varphi = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \beta = \frac{2}{\sqrt{3}} \]
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\[ h(t) = \frac{2}{\sqrt{3}} e^{\frac{-j\pi}{2}} \cos \left( \frac{\sqrt{3}}{2} t + \frac{\pi}{6} \right) u(t) \]

\( \omega = 0 \quad H(j0) = 0 \)
\( \omega = 1 \quad H(j1) = 1 \quad \varphi(1) = 0 \)
\( \omega = 3 \quad H(j3) = \frac{j3}{-8 + j3} = 0.351e^{-j69.44^\circ} \)
\( \omega = 5 \quad H(j5) = \frac{j5}{-24 + j5} = 0.204e^{-j\theta^\circ} \)

\[ y(t) = \frac{4}{\pi^2} \left[ \cos t + \frac{0.351}{9} \cos(3t - 69.44) + \frac{0.204}{25} \cos(5t - 78.23^\circ) + \cdots \right] \]

\[ y(t) = \frac{4}{\pi^2} \left[ \cos t + 0.039 \cos(3t - 69.44) + 0.00816 \cos(5t - 78.23^\circ) + \cdots \right] \]

Q.7

(a) \[ F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j\omega t} dt = -\frac{1}{j\omega} e^{-j\omega t} \bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = -\frac{1}{j\omega} \left[ e^{-j\omega \frac{\tau}{2}} - e^{j\omega \frac{\tau}{2}} \right] \]
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\[ F(j \omega) = \tau \frac{\sin\left(\frac{\omega \tau}{2}\right)}{\frac{\omega \tau}{2}} \]

\[ C_k = \frac{1}{T_o} F(j k \omega_o) \quad \text{where} \quad \omega_o = \frac{2\pi}{T_o} \]

\[ C_k = \frac{\tau}{T_o} \frac{\sin\left(\frac{k \pi \tau}{T_o}\right)}{\frac{k \pi \tau}{T_o}} \]

(b) If \( \frac{T_o}{\tau} = N \) then we have,

\[ C_k = \frac{1}{N} \sin\left(\frac{k \pi}{N}\right) \]

Since \( m = \pm 1, \pm 2, \pm 3, \ldots \) and \( k = mN \) \Rightarrow \( C_k = \frac{1}{N} \sin\left(m\pi\right) = 0 \)

\[
\begin{array}{c|c|c|c|c}
 k & 1 & 2 & 3 & 4 \\
\hline
 C_k \quad \frac{\sqrt{3}}{\pi} & \frac{\sqrt{3}}{2\pi} & 0 & -\frac{\sqrt{3}}{4\pi} \\
|C_k| \quad \frac{\sqrt{3}}{\pi} & \frac{\sqrt{3}}{2\pi} & 0 & \frac{\sqrt{3}}{4\pi} \\
\phi_k \quad 0 & 0 & -\pi &
\end{array}
\]
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\[ f_p(t) = \sum_{k=0}^{\infty} \alpha_k \cos(\omega_k t + \phi_k); \quad \begin{cases} 
\alpha_o = C_o \\
\alpha_k = 2|C_k| \\
\alpha_k = \arg\{C_k\} 
\end{cases} \quad \text{for } k \geq 1 \]

\[ f_p(t) = \frac{1 + \sqrt{3}}{\pi} \left[ \cos(\omega t) + \frac{1}{2} \cos(2\omega t) + 0 - \frac{1}{4} \cos(4\omega t) \right] \]

(c)

\[ g_p(t) = 2f_p(t - \frac{\pi}{2}) - 1 \]

\[ f_p(t) \Rightarrow C_k \]

\[ f_p\left(t - \frac{\pi}{2}\right) \Rightarrow C_k e^{-\frac{j2\pi}{3}} = C_k e^{-j\frac{k\pi}{3}} \]

“−1” only affects \( C_o \) as it is a DC term. Therefore

\[ C^p_o = 2\left(\frac{1}{3} - 1\right) = -\frac{1}{3} \]

For \( g_p \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>C_k</td>
<td>)</td>
<td>( \frac{2\sqrt{3}}{\pi} )</td>
<td>( \frac{\sqrt{3}}{\pi} )</td>
</tr>
<tr>
<td>( \varphi_k )</td>
<td>( -\frac{\sqrt{3}}{\pi} )</td>
<td>( -\frac{2\sqrt{3}}{\pi} )</td>
<td>( \pi - \frac{4\pi}{3} ) = ( -\frac{\pi}{3} )</td>
<td></td>
</tr>
</tbody>
</table>

\[ g_p(t) = -\frac{1 + 2\sqrt{3}}{3} \left[ \cos\left(\omega t - \frac{\pi}{3}\right) + \frac{1}{2} \cos\left(2\omega t - \frac{2\pi}{3}\right) + 0 + \frac{1}{4} \cos\left(4\omega t - \frac{\pi}{3}\right) + \cdots \right] \]

Q.8 (a) Given:

\[ v_1(t) - v_2(t) = \frac{dv_1(t)}{dt} + \int_0^t v_2(t) dt \]

Differentiating both sides, we get

\[ \frac{dv_1(t)}{dt} - \frac{dv_2(t)}{dt} = \frac{d^2v_1(t)}{dt^2} + v_2(t) \]

The Fourier transform of both sides yields

\[ j\omega V_1(j\omega) - j\omega V_2(j\omega) = -\omega^2 V_1(j\omega) + V_2(j\omega) \]
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\[ H(j\Omega) = \frac{V_2(j\omega)}{V_1(j\omega)} = \frac{j\omega}{1-\omega^2 + j\omega} \]

\[ |H(j\omega)| = \frac{\omega}{\sqrt{(1-\omega^2)^2 + \omega^2}} = \frac{1}{\sqrt{1 + \left(\frac{\omega - 1}{\omega}\right)^2}} \]

\[ \mathcal{F}\left\{ e^{-at} \cos(\omega t + \varphi) u(t) \right\} = \frac{\alpha + j\omega - \omega_o \tan \varphi}{(\alpha + j\omega)^2 + \omega_o^2} = \frac{\alpha + j\omega - \omega_o \tan \varphi}{\alpha^2 + \omega_o^2 - \omega^2 + j2\alpha \omega} \]

(b) Dividing both sides by \( \cos \varphi \) we get

\[ \frac{1}{\cos \varphi} \mathcal{F}\left\{ e^{-at} \cos(\omega t + \varphi) u(t) \right\} = \frac{\alpha + j\omega - \omega_o \tan \varphi}{(\alpha + j\omega)^2 + \omega_o^2} \]

Setting

\[ \alpha^2 + \omega_o^2 = 1, \quad 2\alpha = 1, \quad \tan \varphi = \frac{\alpha}{\omega_o} \]

we obtain

\[ \alpha = \frac{1}{\sqrt{2}}, \quad \omega_o = \frac{1}{\sqrt{2}} \quad \text{and} \quad \varphi = \frac{\pi}{4} \]

Therefore

\[ \mathcal{F}\left\{ e^{-t/2} \cos\left(\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right) u(t) \right\} = \frac{j\omega}{1-\omega_o^2 + j\omega} = H(j\omega) \]

hence

\[ h(t) = \sqrt{2}e^{-t/2} \cos\left(\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right) u(t) \]

Q.9 (a)

\( x(t) \) can be written as

\[ x(t) = p\left(t + \frac{a}{2}\right) - p\left(t - \frac{a}{2}\right) \]
The Fourier transform of $x(t)$ yields

$$X(j\omega) = e^{j\frac{a\omega}{2}}P(j\omega) - e^{-j\frac{a\omega}{2}}P(j\omega) = 4e^{j\frac{a\omega}{2}} - e^{-j\frac{a\omega}{2}} \frac{\sin\left(\frac{a\omega}{2}\right)}{\omega}$$

hence

$$X(j\omega) = 4j\sin\left(\frac{a\omega}{2}\right) \left(\frac{\sin\left(\frac{a\omega}{2}\right)}{\omega}\right) = 4j^2 \left(\sin\frac{a\omega}{2}\right)^2$$

(b) From part (a), we can write

$$X(0) = e^{j0}P(0) - e^{-j0}P(0) = 0$$

Therefore from

$$X(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi\delta(\omega) X(0)$$

we obtain

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega)$$

Now using the result of part (a), we get

$$Y(j\omega) = \frac{1}{j\omega} X(j\omega) = 4\left(\frac{\sin\left(\frac{a\omega}{2}\right)}{\omega}\right) = \left(\frac{\sin\frac{a\omega}{2}}{\omega}\right)^2$$

Q.10 (a) We know that the Fourier coefficients $C_k^x$ and $C_k^y$ of $x_p(t)$ and $y_p(t)$ are given as:

$$C_k^x = \frac{1}{T_0} X(jk\omega_o)$$

$$C_k^y = \frac{1}{T_0} Y(jk\omega_o)$$

The fundamental period $T_o$ of $x_p(t)$ and $y_p(t)$ is $2a$. Therefore, the fundamental frequency (frequency of the first harmonic) is given as: $F_o = \frac{1}{T_0} = \frac{1}{2a}$ which yields

$$\omega_o = 2\pi F_o = \frac{\pi}{a}$$

Using this we obtain
Let us now find $C^y_k$:

$$C^y_k = \frac{a^2}{2a} \left( \frac{\sin k \pi}{2} \right)^2 = \frac{1}{2} \frac{\sin^2 k \pi}{a^2}$$

For:

- $k$ even $\left(\sin^2 \frac{k \pi}{2} = 0\right)$
- $k$ odd $\left(\sin^2 \frac{k \pi}{2} = 1\right)$

(b) According to the integration-in-time property the relationship between the Fourier transforms of $x(t)$ and $y(t)$ is given as:

$$X(j\omega) = j\omega Y(j\omega)$$

Substituting $\omega = k\frac{\pi}{a}$ and dividing both sides with $T_0$, we obtain

$$\frac{1}{T_0} X\left( jk \frac{\pi}{a} \right) = jk \frac{\pi}{a} \frac{1}{T_0} Y\left( jk \frac{\pi}{a} \right)$$

Now considering that

$$C^x_k = \frac{1}{T_0} X\left( jk \frac{\pi}{a} \right)$$
$$C^y_k = \frac{1}{T_0} Y\left( jk \frac{\pi}{a} \right)$$

the relationship between $C^x_k$ and $C^y_k$ is obtained as:

$$C^x_k = jk \frac{\pi}{a} C^y_k$$

Taking $a = 1$, we obtain
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\[ C_y^k = \frac{1}{2} \left( \frac{\sin k\pi}{2k} \right)^2 \quad \text{and} \quad C_x^k = \frac{2}{\pi} \frac{(\sin k\pi/2)^2}{k} \]

c\_y^k \text{ and } \Im\{c\_x^k\} \text{ are sketched versus } k \text{ below:}

\[ y_p(t) = \frac{2}{\pi^2} \sum_{k=-\infty}^{\infty} \left( \frac{\sin k\pi/2}{k} \right)^2 \sin(\pi k t) \]

where

\[ \frac{2}{\pi^2} \left( \frac{\sin k\pi/2}{k} \right)^2 = \begin{cases} 
\frac{2}{(k\pi)^2} & \text{for } k \text{ odd} \\
0 & \text{for } k \text{ even} \\
\frac{1}{2} & \text{for } k = 0 
\end{cases} \]

Therefore for \( k > 0 \), we can replace \( k \) with \( 2k' - 1 \); and for \( k < 0 \) with \( 2k'' - 1 \), hence:
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\[ y_p(t) = \frac{2}{\pi^2} \sum_{k=-\infty}^{1} \frac{1}{(2k+1)^2} e^{j(2k+1)\pi t} + \frac{1}{2} + \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k)^2} e^{j(2k-1)\pi t} \]

which yields

\[ y_p(t) = \frac{1}{2} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)\pi t \]

or

\[ y_p(t) = \frac{1}{2} + \frac{4}{\pi^2} \left( \cos \pi t + \frac{1}{9} \cos 3\pi t + \frac{1}{25} \cos 5\pi t + \cdots \right) \]

\[ y_p(t) = \sum_{k=-\infty}^{\infty} C_k^y e^{j\pi t} \]

\[ \frac{dy_p(t)}{dt} = \sum_{k=-\infty}^{\infty} jk \pi C_k^y e^{j\pi t} \]

Considering the integration-in-time property between \( x_p(t) \) and \( y_p(t) \) given above as

\[ C_k^x = jk \pi C_k^y \]

we can write

\[ \frac{dy_p(t)}{dt} = \sum_{k=-\infty}^{\infty} C_k^y e^{j\pi t} = x_p(t) \quad \text{Q.E.D.} \]

**Q.11 (a)** The waveform in Fig.9b can be given as:

\[ g(t) = -\frac{df(t)}{dt} \]

Therefore its F.T. is:

\[ G(j\omega) = -j\omega F(j\omega) = -j\omega \left( \frac{\sin \omega}{2} \right)^2 \]

On the other hand the waveform in Fig.9c can be given as:

\[ g(t + 0.5) \]

whose F.T. is given as

\[ \mathcal{F}\{g(t + 0.5)\} = e^{j0.5\Omega} G(j\omega) = -je^{j0.5\Omega} \left( \frac{\sin \omega}{2} \right)^2 \]

The waveform in Fig.9d is given with the function:

\[ -g(t - 0.5) \]

whose F.T. is computed as:
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\[ \mathcal{F} \left\{ -g(t - 0.5) \right\} = je^{-j0.5\omega} \left\{ \sin \left( \frac{\omega}{2} \right) \right\} \]

(b)

\[ \mathcal{F} \left\{ f(t) \cos \omega_d t \right\} = \frac{1}{2} F(\omega - \omega_d) + \frac{1}{2} F(\omega + \omega_d) = \frac{1}{2} \sin \left( \frac{\omega - \omega_d}{2} \right) + \frac{1}{2} \sin \left( \frac{\omega + \omega_d}{2} \right) \]

Q.12

(a) The block diagram for the difference equation

\[ y(n) = 1.08y(n-1) + K\delta(n) - x(n) \]

is given below.

(b) The difference equation given above has two inputs: (i) \( K\delta(t) \) and (ii) \(-x(n) = -r\delta(n-1) - r\delta(n-2) - r\delta(n-3) - r\delta(n-4) \). Therefore we can first find the response to \( K\delta(t) \), \( y_1(t) \), and then the response to \(-x(n) \), \( y_2(t) \) and add the two to obtain the complete response, \( y(t) = y_1(t) + y_2(t) \). In fact, the response to \( K\delta(t) \) will be \( Kx( \text{the impulse response } h(t) ) \) which can be found as follows:

Since for \( \delta(t) \) as the input the output is given as \( y(t) = h(t) \), the impulse response, then we can write from the above difference equation
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\( h(n) = 1.08h(n-1) + \delta(t) \)

which yields

\[
\begin{align*}
  n = 0 & \quad h(0) = 1.08 h(-1) + 1 = 1 & \text{as} & \quad h(-1) = y(-1) = 0 \\
  n \geq 1 & \quad h(n) = 1.08 h(n-1)
\end{align*}
\]

As \( \delta(n) \) disappears for \( n > 0 \) we will have

\[
\begin{align*}
  n = 1 & \quad h(1) = 1.08 h(0) \\
  n = 2 & \quad h(2) = 1.08 h(1) = 1.08^2 h(0) \\
  \vdots & \quad \vdots \\
  n = k & \quad h(k) = 1.08^k h(0) = 1.08^k
\end{align*}
\]

It follows from the above that

\[
\begin{align*}
  h(n) = 1.08^n u(n) \\
  y_1(n) = K h(n) = K 1.08^n u(n)
\end{align*}
\]

Now using \(-x(n) = -r\delta(n-1) - r\delta(n-2) - r\delta(n-3) - r\delta(n-4)\) as the input the difference equation for this input is given as:

\[
y_2(n) = 1.08 y_2(n-1) - x(n)
\]

it is evident that

\[
y_2(n) = -r h(n-1) - r h(n-2) - r h(n-3) - r h(n-4)
\]

Hence

\[
y(n) = y_1(n) + y_2(n) = K 1.08^n - r \left( 1.08^{n-1} u(n-1) + 1.08^{n-2} u(n-2) + 1.08^{n-3} u(n-3) + 1.08^{n-4} u(n-4) \right)
\]

For we should have \( n = 4 \)

\[
y(4) = y_1(4) + y_2(4) = 0
\]

For \( n = 4 \)

\[
y_1(4) = K 1.08^4
\]

\[
y_2(4) = -r \left( 1.08^3 + 1.08^2 + 1.08 + 1 \right)
\]

\[
1.36K - 4.506r = 0 \quad r = \frac{1.3604 \times 10^4}{4.506} = 3.019
\]

Now that for \( n = 4 \)

\[
y(4) = 0 \quad \text{we should show that}
\]

\[
y(n) = y_1(n) + y_2(n) \quad \text{for } n \geq 5
\]

What we have proved above is that for \( n = 4 \)

\[
y_2(4) = -y_1(4)
\]
We also know that for \( n \geq 5 \) \( y_1(n) = K 1.08^n u(n) \) will still be valid. However, 
\( y_2(n) \) will have to be recalculated as the input causing \( y_2(n) \) has become zero 
for \( n \geq 5 \). This is easily done as the difference equation for \( n \geq 5 \) is obtained as

\[
y_2(n) = 1.08 y_2(n-1)
\]

for \( n \geq 5 \) with the initial condition

\[
y_2(4) = -y_1(4) = -K 1.08^4
\]

Since the solution of this difference equation is given as

\[
y_2(5) = 1.08 y_2(4) = -1.08^5 K
\]

and for \( n \geq 5 \) we have

\[
y_2(n) = -1.08^n K
\]

It is obvious that

\[
y(n) = y_1(n) + y_2(n) = 0 \quad \text{for} \quad n \geq 5
\]

Q.E.D.

\[
Q.13
\]

\[
y(n) = \frac{\sqrt{3}}{2} y(n-1) - \frac{1}{4} y(n-2) + x(n) - x(n-2)
\]

For the impulse response we take

\[
x(n) = \delta(n) \text{ and } y(-1) = y(-2) = 0
\]
which yields

\[ n = 0 \Rightarrow y(0) = 1 \]

\[ n = 1 \Rightarrow y(1) = \frac{\sqrt{3}}{2}, \quad y(0) = \frac{\sqrt{3}}{2} = 0.866 \]

\[ n = 2 \Rightarrow y(2) = \frac{\sqrt{3}}{2} y(1) - \frac{1}{4} y(0) - 1 = \frac{3}{4} - \frac{1}{4} - 1 = -\frac{1}{2} \]

\[ n = 3 \Rightarrow y(3) = \frac{\sqrt{3}}{2} y(2) - \frac{1}{4} y(1) = -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{8} = -0.6495 \]

\[ n = 4 \Rightarrow y(4) = \frac{\sqrt{3}}{2} y(3) - \frac{1}{4} y(2) = -\frac{9}{16} + \frac{1}{8} = -0.4375 \]

\[ n = 5 \Rightarrow y(5) = \frac{\sqrt{3}}{2} y(4) - \frac{1}{4} y(3) = -\frac{7\sqrt{3}}{32} + \frac{3\sqrt{3}}{32} = -0.2165 \]

Q.19

(a) \( h(t) = e^{-\alpha t} u(t); \quad \alpha > 0 \)

\[ \mathcal{Z}\{h(t)\} = H(j\omega) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) e^{-j\omega t} \, dt \]

\[ = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} \, dt = \int_{0}^{\infty} e^{-\omega t} e^{-j\omega t} \, dt \]

\[ = \frac{1}{-(\alpha + j\omega)} e^{-\omega t} \bigg|_{0}^{\infty} = e^{-\alpha t} e^{-j\omega t} \bigg|_{0}^{\infty} \]
Since for $t = \infty$ $e^{-\alpha t} = 0$, we obtain

\[ H(j\omega) = \frac{1}{\alpha + j\omega} = \frac{1}{\sqrt{\alpha^2 + \omega^2}} e^{-j\tan^{-1}\frac{\omega}{\alpha}} \]

\[ |H(j\omega)| = \frac{1}{\sqrt{\alpha^2 + \omega^2}} \]

\[ |H(0)| = \frac{1}{\alpha}, \quad |H(j\alpha)| = \frac{1}{\sqrt{2}\alpha} \]

hence

\[ |H(0)| = \frac{1}{\alpha}, \quad \frac{|H(j\alpha)|}{|H(0)|} = \frac{1}{\sqrt{2}} \]

We can conclude that the 3-dB cutoff frequency and the 3-dB bandwidth are given by

\[ \omega_c = W_{3\text{dB}} = \alpha \]

(b) \[
\mathcal{F}\{h(t)\cos\omega_b t\} = \frac{1}{2} H(j(\omega - \omega_b)) + \frac{1}{2} H(j(\omega + \omega_b))
\]

\[
\mathcal{F}\{e^{-\alpha t}u(t)\cos\omega_b t\} = \frac{1/2}{\alpha + j(\omega - \omega_b)} + \frac{1/2}{\alpha + j(\omega + \omega_b)} = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_b^2} = \tilde{H}(j\omega)
\]
A measure for testing whether $\hat{H}(j\omega)$ represents a bandpass filter is

$$\left| \frac{\hat{H}(j\omega_b)}{H(0)} \right| \gg 1$$

For $\omega = 0$

$$\hat{H}(0) = \frac{\alpha}{\alpha^2 + \omega_b^2}$$

For $\omega = \omega_b$

$$\hat{H}(j\omega_b) = \frac{\alpha + j\omega_b}{(\alpha + j\omega_b)^2 + \omega_b^2} = \frac{\alpha + j\omega_b}{\alpha^2 + j2\alpha\omega_b}$$

and for $\omega = -\omega_b$

$$\hat{H}(-j\omega_b) = \frac{\alpha - j\omega_b}{(\alpha - j\omega_b)^2 + \omega_b^2} = \frac{\alpha - j\omega_b}{\alpha^2 - j2\alpha\omega_b} = \bar{\hat{H}}(j\omega_b)$$

$$|\hat{H}(-j\omega_b)| = |\hat{H}(j\omega_b)| = \sqrt{\frac{\alpha^2 + \omega_b^2}{\alpha^2 + 4\alpha^2\omega_b^2}}$$
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\[
\left| \frac{H(j\omega_b)}{H(0)} \right| = \sqrt{\frac{\omega_c^2 + \omega_b^2}{\omega_c^4 + 4\omega_c^2\omega_b^2}} \approx \frac{1}{2} \left( \frac{\omega_b}{\omega_c} \right)^2
\]

Since \( \alpha = \omega_c \)

\[
\frac{\omega_c}{\omega_b} \ll 1
\]

implies that

\[
\left| \frac{H(j\omega_b)}{H(0)} \right| \gg 1
\]

hence \( H(j\omega) \) represents a bandpass filter.

Q.21

(a)

\[
Y^+(z) = \sum_{n=0}^{\infty} y(n)z^{-n} = \sum_{n=-\infty}^{\infty} y(n)u(n)z^{-n} = \mathcal{Z} \{ y(n)u(n) \} \quad \text{Q.E.D.}
\]

(b) One-sided z-transform of \( y(n) = \alpha y(n-1) + x(n) \) yields

\[
Y^+(z) = az^{-1} \left[ Y^+(z) + y(-1)z \right] + X^+(z)
\]

and we obtain

\[
Y^+(z) = \frac{\alpha}{1-\alpha z^{-1}} y(-1) + \frac{X^+(z)}{1-\alpha z^{-1}} = \frac{\alpha z}{z-\alpha} y(-1) + \frac{z}{z-\alpha} X^+(z) = Y^+_1(z) + Y^+_2(z)
\]

Since \( x(n) = u(n) \Rightarrow X(z) = \frac{1}{1-z^{-1}} \) and the above yields

\[
Y^+(z) = \frac{\alpha}{1-\alpha z^{-1}} y(-1) + \frac{X^+(z)}{1-\alpha z^{-1}} = \frac{\alpha}{1-\alpha z^{-1}} y(-1) + \frac{1}{(1-\alpha z^{-1})(1-z^{-1})} = \frac{\alpha z}{z-\alpha} y(-1) + \frac{z^2}{(z-\alpha)(z-1)}
\]
Expanding $\frac{Y^*_2(z)}{z}$ into partial fractions, we obtain

$$\frac{Y^*_2(z)}{z} = \frac{z}{(z-\alpha)(z-1)} = \frac{k_1}{z-\alpha} + \frac{k_2}{z-1}$$

where

$$k_1 = \left. \frac{Y^*_2(z)}{z}(z-\alpha) \right|_{z=\alpha} = \frac{z}{z-1} = \frac{\alpha}{\alpha - 1} = \frac{-\alpha}{1-\alpha}$$

$$k_2 = \left. \frac{Y^*_2(z)}{z}(z-1) \right|_{z=1} = \frac{1}{1-\alpha}$$

Now we can write

$$Y^*_2(z) = \frac{-\alpha}{1-\alpha} \frac{z}{z-\alpha} + \frac{1}{1-\alpha} \frac{z}{z-1} = \frac{-\alpha}{1-\alpha} \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha} \frac{1}{1-z^{-1}}$$

Taking the inverse transform of both sides gives

$$y_2(n) = \frac{-\alpha}{1-\alpha} \alpha^n u(n) + \frac{1}{1-\alpha} u(n)$$

On the other hand we have

$$y_1(n) = z^{-1} \{Y^*_1(z)\} = z^{-1} \left\{ \frac{\alpha y(-1)}{1-\alpha z^{-1}} \right\} = \alpha y(-1) \alpha^n u(n)$$

and finally the complete response is obtained as:

$$y(n) = y_{2ir}(n) + y_{2ir}(n) = \alpha y(-1) \alpha^n u(n) + \frac{-\alpha}{1-\alpha} \alpha^n u(n) + \frac{1}{1-\alpha} u(n)$$

$$y(n) = \alpha^{n+1} u(n) + \frac{1-\alpha^{n+1}}{1-\alpha} u(n)$$

(c) The complete response is given as:

$$Y^*(z) = \frac{\alpha}{1-\alpha z^{-1}} y(-1) + \frac{X^+(z)}{1-\alpha z^{-1}} = \frac{\alpha z}{z-\alpha} y(-1) + \frac{z}{z-\alpha} X^+(z) = Y^*_1(z) + Y^*_2(z)$$

hence

$$y(n) = Z^{-j} \{Y^*(z)\} = Z^{-j} \{Y^*_1(z)\} + Z^{-j} \{Y^*_2(z)\} = y_1(n) + y_2(n)$$

where the first term is due to the initial value of the state, $y(-1)$, and the second term is due to the input, $x(n)$. Evidently, $y_1(n)$ is obtained from $y(n)$ when the input is is zero and $y_2(n)$ is obtained from $y(n)$ when the initial value of the state, $y(-1)$, is zero, that is:
y_1(n) = y(n)|_{y(n) \neq 0} \quad , \quad y_2(n) = y(n)|_{y(n) \neq -1}

Henceforth \( y_1(n) \) and \( y_2(n) \) will be called the **Zero-Input Response** and the **Zero-State Response**, respectively:

\[
y_1(n) = y_{zir}(n) \quad \text{and} \quad y_2(n) = y_{zsr}(n)
\]

Therefore

\[
y(n) = z^{-1}\{Y^+(z)\} = y_{zir}(n) + y_{zsr}(n)
\]

i.e.,

**Complete response (CR)= Zero-input response (ZIR) + Zero-state response (ZSR)**

Now considering

\[
Y_{zir}^+(z) = \frac{az}{z - \alpha} y(-1)
\]

and

\[
Y_{zsr}^+(z) = \frac{\alpha}{\alpha - 1} \frac{z}{z - \alpha} + \frac{1}{1 - \alpha} \frac{z}{z - 1} = Y_{zir}^+(z) + Y_{zsr}^+(z)
\]

we see that \( Y_{zir}^+(z) \) and \( Y_{zsr}^+(z) \) have the same denominator which is the denominator of the transfer function and gives rise to same type of response. On the other hand, the denominator of \( Y_{zsr}^+(z) \) is the same as that of the \( z \)-transform of the input. Therefore we can regroup the partial fractions according to their denominators which yields:

\[
y(n) = \begin{cases} Y_{zir}^+(n) + Y_{zsr}^+(n) & \text{Natural Response} \\ Y_{zsr}^+(n) & \text{Forced Response} \end{cases}
\]

The complete response is also given as:

**Complete response = Natural response (NR) + Forced response (FR)**

We also know that NR is due to the roots of the characteristic polynomial, i.e., the denominator of the transfer function and FR is due to the denominator of the \( z \)-transform of the input.

Now considering

\[
Y_1^+(z) = \frac{\alpha y(-1)z}{z - \alpha}
\]

and
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\[ Y_2^+(z) = Y_{21}^+(z) + Y_{22}^+(z) = \frac{\alpha}{\alpha - 1} \frac{z}{z - \alpha} + \frac{1}{1 - \alpha} \frac{z}{z - 1} \]
\[ Y_{2r}^+(z) = \frac{-\alpha}{1 - \alpha} \frac{z}{z - \alpha} + \frac{1}{1 - \alpha} \frac{z}{1 - z^{-1}} = Y_{2r_1}^+(z) + Y_{2r_2}^+(z) \]

we see that \( Y_1^+(z) \) and \( Y_{2r}^+(z) \) have the same denominator which is the denominator of the transfer function and gives rise to same type of response. On the other hand, the denominator of \( Y_{2r}^+(z) = \frac{1}{1 - \alpha} \frac{z}{z - 1} \) is the same as that of the z-transform of the input, hence the following:

\[
\begin{align*}
\text{Natural Response} & \quad \text{Forced Response} \\
y_2(n) &= y_1(n) + y_{21}(n) + \underbrace{y_{21}(n)}_\text{Zero-state response} \\
y_2(n) &= \underbrace{y_1(n)}_\text{Zero-input response} + y_{21}(n) + y_{21}(n)
\end{align*}
\]

<table>
<thead>
<tr>
<th>Type of response</th>
<th>Expression</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero-input response</td>
<td>( y_1(n) = y(-1)\alpha^{n+1}u(n) )</td>
<td>The part of CR originating from the initial state value only, i.e., ( y(n) = y_1(n) ) ( \big</td>
</tr>
<tr>
<td>Zero-state response</td>
<td>( y_2(n) = \frac{\alpha}{1 - \alpha} \frac{z}{z - \alpha} u(n) )</td>
<td>The part of CR originating from the input only, i.e., ( y(n) = y_2(n) ) ( \big</td>
</tr>
<tr>
<td>Natural response</td>
<td>( \alpha y(-1)\alpha^u(n) + \frac{\alpha}{\alpha - 1} u(n) )</td>
<td>The part of CR originating from the denominator of the transfer function</td>
</tr>
<tr>
<td>Forced response</td>
<td>( \frac{1}{1 - \alpha} u(n) )</td>
<td>The part of CR originating from the denominator of the z-transform of input</td>
</tr>
</tbody>
</table>

\[
Y^+(z) = \frac{\alpha}{1 - \alpha z^{-1}} y(-1) + \frac{X^+(z)}{1 - \alpha z^{-1}} = \frac{\alpha}{1 - \alpha z^{-1}} y(-1) + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})} = \\
= \frac{\alpha y(-1)z}{z - \alpha} + \frac{z^2}{z - \alpha}(z - 1) = Y_{2r_1}^+(z) + Y_{2r_2}^+(z)
\]

\[
Y^+(z) = \frac{\alpha}{1 - \alpha z^{-1}} y(-1) + \frac{X^+(z)}{1 - \alpha z^{-1}} = \frac{\alpha}{1 - \alpha z^{-1}} y(-1) + \frac{1}{(1 - \alpha z^{-1})(1 - z^{-1})} = \\
= \frac{\alpha z}{z - \alpha} y(-1) + \frac{z^2}{z - \alpha}(z - 1) = Y_1^+(z) + Y_2^+(z)
\]
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hen

\[ Y^{+}_{zsr}(z) = -\alpha \frac{z}{1 - \alpha z - \alpha} + \frac{1}{1 - \alpha} \frac{z}{z-1} = -\alpha \frac{1}{1 - \alpha} + \frac{1}{1 - \alpha} = Y^{+}_{zsr_1}(z) + Y^{+}_{zsr_2}(z) \]

hence

\[ y(n) = Z^{-1}\{Y^{+}(z)\} = Z^{-1}\{Y^{+}_{zsr_1}(z)\} + Z^{-1}\{Y^{+}_{zsr_2}(z)\} = y_1(n) + y_2(n) \]

where the first term is due to the initial value of the state, \( y(-1) \), and the second term is due to the input, \( x(n) \). Evidently, \( y_1(n) \) is obtained from \( y(n) \) when the input is is zero and \( y_2(n) \) is obtained from \( y(n) \) when the initial value of the state, \( y(-1) \), is zero, that is:

\[ y_1(n) = y(n)|_{x(n)=0}, \quad y_2(n) = y(n)|_{y(-1)=0} \]

Henceforth \( y_1(n) \) and \( y_2(n) \) will be called the **Zero-Input Response** and the **Zero-State Response**, respectively:

\[ y_{zsr}(n) = y_{zsr_1}(n) + y_{zsr_2}(n) = -\alpha \frac{\alpha^n u(n)}{1 - \alpha} + \frac{1}{1 - \alpha} u(n) \]