SAMPLING OF 1-D SIGNALS AND IMAGES

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Sampling

\[ x_a(t) = A \cos(\Omega_0 t + \Phi) \]

Sampling of \( x_a(t) \) implies the substitution:

\[ t = nT_s; \quad n : \text{integer} \]

which yields:

\[ x_a(nT_s) = A \cos(\Omega_0 T_s n + \Phi) \]
Sampling

Since

\[ \frac{1}{T_s} = F_s \]

\[ x_a(nT_s) = A \cos \left( \frac{\Omega_0}{F_s} n + \Phi \right) \]

\[ = A \cos \left( 2\pi \frac{F_0}{F_s} n + \Phi \right) \]
Sampling

Defining:

\[ x_a(nT_s) = x(n) \]

and

\[ \omega_0 = 2\pi \frac{F_0}{F_s} \]

yields

\[ x(n) = A \cos(\omega_0 n + \Phi) \]
Sampling

Ideal Analogue/Digital Converter (A/D)

\[ x_a(t) = A \cos(2\pi F_0 t + \Phi) \]

\[ x(n) = A \cos(2\pi \frac{F_0}{F_s} n + \Phi) \]

Ideal Digital/Analogue Converter (D/A)

\[ x(n) = A \cos(2\pi \frac{F_0}{F_s} n + \Phi) \]

\[ x_a(t) = A \cos(2\pi F_0 t + \Phi) \]
Sampling

\[ \Omega_0 = 2\pi F_0, \quad T_s = \frac{1}{F_s} \]

\[ \Omega_0 T_s = 2\pi \frac{F_0}{F_s} = 2\pi f_0 = \omega_0 \]

where

\[ f_0 = \frac{F_0}{F_s} \]

is called either of the following:

- Normalised frequency
- Relative frequency
- Discrete-time frequency
- Digital frequency
Sampling

Consider

\[ f_0 = \frac{F_0}{F_s} \]

whose dimension is given by:

\[ \frac{\text{cycles/s}}{\text{samples/s}} = \frac{\text{cycles}}{\text{sample}} \]

which reveals that \( f_0 \) is the number of cycles per sample.
Sampling

Now consider

\[ N_0 = \frac{1}{f_0} = \frac{F_s}{F_0} \]

whose dimension is given by:

\[ \frac{\text{samples}/s}{\text{cycles}/s} = \frac{\text{samples}}{\text{cycle}} \]

It is obvious that \( N_0 \) can also be given as:

\[ N_0 = \frac{2\pi}{\omega_0} \]
Sampling

and $\omega_0$ is called either of the following:

- Normalised angular frequency
- Relative angular frequency
- Discrete-time angular frequency
- Digital angular frequency
Sampling

The unit and meaning of $\omega_0$:

$$\omega_0 = 2\pi f_0 = \frac{2\pi \text{ radians/cycle}}{N_0 \text{ samples/cycle}} = \frac{2\pi \text{ radians}}{N_0 \text{ sample}}$$

which reveals that $\omega_0$ gives the portion of angle in radians that lies between two samples:
Sampling

Example:

\[ A = 1, \ F_0 = 30 \text{Hz}, \ F_s = 12F_0 = 360 \text{Hz}, \ \Phi = 0 \]

\[ \omega_0 = 2\pi f_0 = 2\pi \frac{F_0}{F_s} = 2\pi \frac{30}{360} = 2\pi \frac{1}{12} = \frac{\pi}{6} \]

where

\[ f_0 = \frac{1}{12} \quad \text{and} \quad N_0 = \frac{1}{f_0} = 12 \]
Sampling

For example

\[ f_0 = \frac{1}{12} \text{ cycle/sample} \]

this means that there is one cycle every 12 samples or we could also say: There is 1/12 cycle per sample.
Sampling

t=0:1/360:1;
x=sin(2*pi*30*t);
stem(x(1:15))
title('x (n)=sin(pi/6)(n-1) ')
Sampling

t=0:1/360:1;
x=sin(2*pi*30*(t+1/360));
stem(x(1:15))
title('x(n)=sin(pi/6)n')
Let us now consider

\[ x(n) = A \cos(\omega_0 n + \Phi + 2kn\pi) \]

\[ = A \cos(\omega_0 n + 2kn\pi + \Phi) \]

\[ = A \cos([\omega_0 + 2k\pi] n + \Phi) \]

\[ = A \cos(\omega_0 n + \Phi) \]
Sampling

An Implication of Sampling

The frequency spectrum of $x(n)$ is periodic with a period of $2\pi$.

Case 1:

$\omega_0 < \pi$
Sampling

\[ x(n) = A \cos(\omega_0 n + \Phi) \]

\[ = A \cos((\omega_0 + 2k\pi)n + \Phi) \]

\[ = A \cos(2\pi \frac{F_0}{F_s} + 2k\pi)n + \Phi) \]

\[ = A \cos(2\pi \frac{F_0 + kF_s}{F_s}n + \Phi) \]
Sampling

Now sampling of

\[ x_a(t) = A \cos(2\pi \left[ F_0 + kF_s \right] t + \Phi) \]

yields:

\[ x_a(nT_s) = A \cos(2\pi \left[ \frac{F_0 + kF_s}{F_s} \right] n + \Phi) \]
Sampling

We obtain

\[ x_a(nT_s) = A \cos(\left(\frac{F_0}{F_s} + 2\pi k \right)n + \Phi) \]

\[ = A \cos(2\pi \frac{F_0}{F_s} + 2\pi kn + \Phi) \]

\[ = A \cos(2\pi \frac{F_0}{F_s} n + \Phi) \]
Sampling

Ideal Analogue/Digital Converter (A/D)

\[ x_a(t) = A\cos(2\pi [F_0 + kF_s]t + \Phi) \quad \rightarrow \quad x(n) = A\cos(2\pi \frac{F_0}{F_s} n + \Phi) \]

Ideal Digital/Analogue Converter (D/A)

\[ x(n) = A\cos(2\pi \frac{F_0}{F_s} n + \Phi) \quad \rightarrow \quad x_a(t) = A\cos(2\pi F_0 t + \Phi) \]
Sampling

Another Implication of Sampling

The frequency $F_0 + kF_s$, where $k$ is an integer, is called an alias of the frequency $F_0$ with respect to the sampling frequency $F_s$, because $F_0 + kF_s$ and $F_0$ appear to be the same when sampled at the rate $F_s$. 
Sampling

Example:

\[ A = 1, F_0 = 30\text{Hz}, F_s = 12F_0 = 360\text{Hz} \]

\[ F'_0 = F_0 + F_s = 13F_0 = 390\text{Hz}, \Phi = 0 \]

\[ x_a(t) = \cos(2\pi[F_0 + F_s]t) = \cos(2\pi390t) \]

\[ T_s = \frac{1}{F_s} = \frac{1}{360} \approx 0.00278\text{s} = 2.78\text{ms} \]
Sampling

\[ x_a(nT_s) = x_a(0.00278n) = x_a\left(\frac{1}{360}n\right) \]

\[ x(n) = \cos 2\pi \frac{390}{360} n = \cos 2\pi \frac{390}{360} n \]

\[ = \cos(2\pi \frac{30}{360} n + 2\pi n) = \cos \frac{\pi}{6} n \]
Sampling

t=0:1/26000:1;
\( y = \sin(2\pi \times 390 \times t); \)
plot(t(1:870),y(1:870))
title('x_\text{a}(t) = \sin(2\pi \times 390 \times t)')
Sampling

t=0:1/360:1;
\[ x = \sin(2\pi \times 30 \times (t+1/360)) \];
figure,stem(x(1:15));
title('x(n)=\sin(\pi/6)n');
\[ x = \sin(2\pi \times 390 \times (t+1/360)) \];
figure,stem(x(1:15));
title('x(n)=\sin(\pi/6+2\pi)n');
Sampling
Sampling

Example:

\[ A = 1, F_0 = 30\, \text{Hz}, F_s = 12F_0 = 360\, \text{Hz} \]

Let \( k = 2 \) in \( F'_0 = F_0 + kF_s \)

\[ F'_0 = F_0 + 2F_s = 25F_0 = 750\, \text{Hz}, \ \Phi = 0 \]

\[ x_a(t) = \cos 2\pi 750t \]
Sampling

t=0:1/50000:1;
y=sin(2*pi*750*t);
plot(t(1:1740),y(1:1740))
title('xa(t)=sin(2pi750t)')
Sampling

t=0:1/360:1;
x=sin(2*pi*30*(t+1/360));
figure,stem(x(1:15));
title('x(n)=sin(pi/6)n');
x=sin(2*pi*750*(t+1/360));
figure,stem(x(1:15));
title('x(n)=sin(pi/6+4pi)n');
Sampling
Now letting $k=0,1$ and $-1$, we have, respectively:

\[ x_a(t) = A \cos(2\pi F_0 t + \Phi) \]
\[ x_{a1}(t) = A \cos(2\pi [F_s + F_0]t + \Phi) \]
\[ x_{a2}(t) = A \cos(2\pi [F_s - F_0]t - \Phi) \]

which give the same samples with the sampling rate $F_s$, i.e.,
Sampling

Note that

\[ x_{a2}(t) = A \cos(2\pi [ F_s - F_0 ] t - \Phi) \]

\[ = A \cos(2\pi [ -F_s + F_0 ] t + \Phi) \]
Sampling

\[ x_a(nT_s) = x(n) \]
\[ x_{a1}(nT_s) = x_1(n) \]
\[ x_{a2}(nT_s) = x_2(n) \]
\[ x(n) = x_1(n) = x_2(n) \]
Sampling

Case 1:

\[ F_0 < \frac{F_s}{2} \]

The frequency spectrum of \( x(n) \) can be given as follows:
Sampling

A/D

D/A

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Sampling

Case 2:

\[ \frac{F_s}{2} < F_0 < F_s \]

There exists an \( F_0' \) such that:

\[ 0 \leq F_0' \leq \frac{F_s}{2} \]

\( F_0 \) can be written as:
Sampling

which is given as:

\[ F'_0 = -F_0 + F_s \quad \text{hence} \quad -F'_0 = F_0 - F_s \]

where obviously

\[ -\frac{F_s}{2} \leq -F'_0 \leq 0 \]

This case corresponds to

\[ k = \pm 1 \]
Sampling

Folding (or pivoting) around $F_s/2$ and $-F_s/2$

A/D

D/A

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Sampling

\[ \Delta F = F_0 - \frac{F_s}{2} \]

Since

\[ \Delta F = \frac{F_s}{2} - (F_s - F_0) = F_0 - \frac{F_s}{2} \]

We can draw the following conclusion:

\[ F_s - F_0 \text{ and } F_0 \]

are mirror images with respect to \( F_s/2 \).
Sampling

Conclusion: Since

\[ \frac{F_s}{2} < F_0 < F_s \]

\( F_0 \) falls outside the region

\[ -\frac{F_s}{2} \leq F \leq \frac{F_s}{2} \]

However, an alias of \( F_0 \), namely \( F_0 - F_s \), and an alias of \(-F_0\), namely \(-F_0 + F_s\), both fall into this region.
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The signal at $F_s - F_0$ is described to be the folded version of $F_0$, although it is an alias of $-F_0$.

The signal at $-F_s + F_0$ is described to be the folded version of $-F_0$, although it is an alias of $F_0$. 
Case 3:

\[ F_s < F_0 \]

This case corresponds to

\[ k = \pm 2, \pm 3, \ldots \]
Sampling

In this case there exists an $F_0'$ such that:

$$0 \leq F_0' \leq \frac{F_s}{2}$$

which is given as:

$$F_0' = -F_0 + |k|F_s \quad \text{hence} \quad -F_0' = F_0 - |k|F_s$$

where

$$k = \pm 2, \pm 3, \ldots \ldots$$
3 Components with frequencies that are outside the range

\[-\frac{F_s}{2} \leq F_0 \leq \frac{F_s}{2}\]

are aliases of those within this range with respect to sampling frequency $F_s$. Having gone through an A/D and D/A they appear as components within this range.
Sampling

Conclusion: In order that no aliasing occur the maximum frequency $F_{\text{max}}$ must satisfy:

$$F_{\text{max}} \leq \frac{F_s}{2}$$
Sampling

Example:

\[ A = 1, F_0 = 240\text{Hz}, F_s = 360\text{Hz}, \Phi = -\frac{\pi}{2} \]

\[ F_0 > \frac{F_s}{2} \]

\[ x_a(t) = \cos\left(2\pi 240t - \frac{\pi}{2}\right) \]

\[ x_a(t) = \sin(2\pi 240t) \]

\[ x(n) = \cos\left(2\pi \frac{240}{360} n - \frac{\pi}{2}\right) = \sin\left(2\pi \frac{240}{360} n\right) \]
Sampling

\[ x(n) = \cos(2\pi \frac{240}{360} n - \frac{\pi}{2}) = \sin(2\pi \frac{240}{360} n) \]

\[ x(n) = \sin(2\pi \frac{240}{360} n) = \frac{1}{2} \sin(2\pi \frac{240}{360} n) - \frac{1}{2} \sin(-2\pi \frac{240}{360} n) \]

\[ \frac{1}{2} \sin(2\pi \frac{240}{360} n - 2\pi n) = \frac{1}{2} \sin(-2\pi \frac{120}{360} n) \]
Sampling

\[
\frac{1}{2} \sin(-2\pi \frac{240}{360} n + 2\pi n) = \frac{1}{2} \sin(2\pi \frac{120}{360} n)
\]

\[
x(n) = \frac{1}{2} \sin(-2\pi \frac{120}{360} n) - \frac{1}{2} \sin(2\pi \frac{120}{360} n)
\]
Sampling

close all

\[ t = 0:1/10000:1; \]
\[ x = \sin(2\pi \cdot 240 \cdot t); \]
\[ \text{plot}(t(1:350), x(1:350)); \]
\[ \text{title}('x_a(t) = \sin(2\pi \cdot 240 \cdot t)'); \]
\[ x = \sin(-2\pi \cdot 120 \cdot t); \]
\[ \text{figure, plot}(t(1:350), x(1:350)); \]
\[ \text{title}('x_a(t) = \sin(-2\pi \cdot 120 \cdot t)'); \]
\[ t = 0:1/360:1; \]
\[ x = \sin(2\pi \cdot 240 \cdot (t+1/360)); \]
\[ \text{figure, stem}(x(1:15)); \]
\[ \text{title}('x(n) = \sin(4\pi / 3) n'); \]
\[ t = 0:1/360:1; \]
\[ x = \sin(-2\pi \cdot 120 \cdot (t+1/360)); \]
\[ \text{figure, stem}(x(1:15)); \]
\[ \text{title}('x(n) = \sin(-2\pi / 3) n'); \]
Sampling
**Sampling**

**Example:** Consider the linear analogue filter given below.

![Diagram](image-url)

- **A/D** input: $x_a(t)$
- **D/A** output: $y_a(t)$
- **Analogue Filter** with frequency $F_s$
- **Digital Filter** with frequency $F_s$
- **Sampling** frequency $\Omega$
- **Angular frequency** $\omega$

**Equation:**

$$\prod_{\text{Digital Filter}} \left( x(n) \right) \xrightarrow{\text{A/D}} \left( x(n) \right) \xrightarrow{\text{Digital Filter}} \left( y(n) \right) \xrightarrow{\text{D/A}} \left( y_a(t) \right)$$
Sampling

The input to the analogue filter is given by:

\[ x_a(t) = \cos(2\pi 3,000t) + \cos(2\pi 6,000t) + \cos(2\pi 40,000t) + \cos(2\pi 60,000t) \]

(a) With \( F_s = 80kHz \), find the component of the input waveform that causes aliasing at this sampling frequency. Whose alias is this frequency? Sketch the frequency spectrum of \( x(n) \) showing how the folding of the alias frequency takes place.

(b) Increase the sampling frequency to \( F_s = 120kHz \) and 240kHz and sketch the frequency spectrum of \( x(n) \).

(c) Sketch the individual components of \( x(n) \).

(d) Take \( y(n) = x(n) \) and express \( y_a(n) \).
Sampling

Solution:

(a) Starting with the highest frequency in $x_a(t)$, calculate the following:

$$f_4 = \frac{F_4}{F_s} = \frac{60,000}{80,000} = \frac{3 \text{ cycles}}{4 \text{ sample}} \Rightarrow \omega_4 = 2\pi f_4 = \frac{3}{2} \pi \text{ radians/sample}$$

$$f_3 = \frac{F_3}{F_s} = \frac{40,000}{80,000} = \frac{1 \text{ cycles}}{2 \text{ sample}} \Rightarrow \omega_3 = 2\pi f_3 = \pi \text{ radians/sample}$$

$F_3$ and therefore $F_2$ and $F_1$ satisfy the Nyquist criterion. The only component that causes aliasing is that with $F_4=60,000$. 
Sampling

-60k -40k -20k 0 20k 40k 60k
-6k -3k 3k 6k

-1/4 -1/2 -1/4 0 1/4 1/2 3/4

-\(\frac{F_s}{2}\) \(\frac{f_s}{2}\)
Sampling

\[ x_a(t) = \cos(2\pi f_0 t) \]

\[ x_a(t) = \cos(2\pi f_0 t) \]
Sampling

\[ x(t) = \cos(2\pi f_0 t) \]

\[ x(t) = \cos(2\pi f_1 t) \]
Sampling
Sampling

\[ x(n) = \cos(\pi n) \]

\[ y(n) = x(n) \cdot \text{Sampling} \]
Sampling

\[ x(n) = \cos\left(\frac{\pi}{2}n\right) \]
Sampling