CHAPTER 8
COMBINATIONS AND PERMUTATIONS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define combinations and permutations.
2. Apply the concept of combinations to problem solving.
3. Apply the concept of principle of choice to problem solving.
4. Apply the concept of permutations to problem solving.

INTRODUCTION

This chapter deals with concepts required for the study of probability and statistics. Statistics is a branch of science that is an outgrowth of the theory of probability. Combinations and permutations are used in both statistics and probability, and they, in turn, involve operations with factorial notation. Therefore, combinations, permutations, and factorial notation are discussed in this chapter.

DEFINITIONS

A combination is defined as a possible selection of a certain number of objects taken from a group without regard to order. For instance, suppose we were to choose two letters from a group of three letters. If the group of three letters were $A$, $B$, and $C$, we could choose the letters in combinations of two as follows:

$AB, AC, BC$
The order in which we wrote the letters is of no concern; that is, \(AB\) could be written \(BA\), but we would still have only one combination of the letters \(A\) and \(B\).

A permutation is defined as a possible selection of a certain number of objects taken from a group with regard to order. The permutations of two letters from the group of three letters would be as follows:

\[AB, AC, BC, BA, CA, CB\]

The symbol used to indicate the foregoing combination will be \(C_2\), meaning a group of three objects taken two at a time. For the previous permutation we will use \(P_2\), meaning a group of three objects taken two at a time and ordered.

You will need an understanding of factorial notation before we begin a detailed discussion of combinations and permutations. We define the product of the integers \(n\) through 1 as \(n\) factorial and use the symbol \(n!\) to denote this; that is,

\[3! = 3 \cdot 2 \cdot 1\]
\[6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\]
\[n! = n \cdot (n - 1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1\]

**EXAMPLE:** Find the value of \(5!\)

**SOLUTION:**

\[5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\]

\[= 120\]

**EXAMPLE:** Find the value of \(\frac{5!}{3!}\)

**SOLUTION:**

\[5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\]

and

\[3! = 3 \cdot 2 \cdot 1\]

Then

\[
\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}
\]
and by simplification
\[
\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 5 \cdot 4
\]
\[
= 20
\]

The previous example could have been solved by writing
\[
\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3!}{3!}
\]
\[
= 5 \cdot 4
\]

Notice that we wrote
\[
5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
\]
and combine the factors
\[
3 \cdot 2 \cdot 1
\]
as
\[
3!
\]
so that
\[
5! = 5 \cdot 4 \cdot 3!
\]

**EXAMPLE:** Find the value of
\[
\frac{6! - 4!}{4!}
\]

**SOLUTION:**
\[
6! = 6 \cdot 5 \cdot 4!
\]

and
\[
4! = 4! \cdot 1
\]

Then
\[
\frac{6! - 4!}{4!} = \frac{(6 \cdot 5 - 1) \cdot 4!}{4!}
\]
\[
= (6 \cdot 5 - 1)
\]
\[
= 29
\]
Notice that $4!$ was factored from the expression

$$6! - 4!$$

Theorem. If $n$ and $r$ are positive integers, with $n$ greater than $r$, then

$$n! = (n)(n - 1) \cdots (r + 2)(r + 1)r!$$

This theorem allows us to simplify an expression as follows:

$$5! = 5 \cdot 4!$$

$$= 5 \cdot 4 \cdot 3!$$

$$= 5 \cdot 4 \cdot 3 \cdot 2!$$

$$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

Another example is

$$(n + 2)! = (n + 2)(n + 1)!$$

$$= (n + 2)(n + 1)(n!)$$

$$= (n + 2)(n + 1)(n) \cdots 1$$

**EXAMPLE:** Simplify

$$\frac{(n + 3)!}{n!}$$

**SOLUTION:**

$$(n + 3)! = (n + 3)(n + 2)(n + 1)n!$$

then

$$\frac{(n + 3)!}{n!} = \frac{(n + 3)(n + 2)(n + 1)n!}{n!}$$

$$= (n + 3)(n + 2)(n + 1)$$
PRACTICE PROBLEMS:

Find the value of problems 1 through 4 and simplify problems 5 and 6.

1. 6!
2. 3!4!
3. \( \frac{8!}{11!} \)
4. \( \frac{5! - 3!}{3!} \)
5. \( \frac{n!}{(n - 1)!} \)
6. \( \frac{(n + 2)!}{n!} \)

ANSWERS:

1. 720
2. 144
3. \( \frac{1}{990} \)
4. 19
5. n
6. \( (n + 1)(n + 2) \)

COMBINATIONS

As indicated previously, a combination is the selection of a certain number of objects taken from a group of objects without regard to order. We use the symbol \( \binom{5}{3} \) to indicate that we have five objects taken three at a time, without regard to order. Using the letters \( A, B, C, D, \) and \( E \) to designate the five objects, we list the combinations as follows:

\[ ABC \ ABD \ ABE \ ACD \ ACE \]
\[ ADE \ BCD \ BCE \ BDE \ CDE \]
We find 10 combinations of 5 objects are taken 3 at a time. The word combinations implies that order is not considered.

**EXAMPLE**: Suppose we wish to know how many color combinations can be made from four different colored marbles if we use only three marbles at a time. The marbles are colored red, green, white, and yellow.

**SOLUTION**: We let the first letter in each word indicate the color; then we list the possible combinations as follows:

\[ \text{RGW RGY RWY GWY} \]

If we rearrange the groups, for example RGW, to form GWR or RWG, we still have the same color combination within each group; therefore, order is not important.

The previous examples are relatively easy to understand; but suppose we have 20 boys and wish to know how many different basketball teams we could form, one at a time, from these boys. Our listing would be quite lengthy, and we would have a difficult task to determine we had all of the possible combinations. In fact, we would have to list over 15,000 combinations. This indicates the need for a formula.

The general formula for possible combinations of \( r \) objects from a group of \( n \) objects is

\[
\binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)}{r \cdots 3 \cdot 2 \cdot 1}
\]

The denominator may be written as

\[
r \cdots 3 \cdot 2 \cdot 1 = r!
\]

and if we multiply both numerator and denominator by

\[
(n - r) \cdots 2 \cdot 1
\]

which is

\[
(n - r)!
\]

we have

\[
\binom{n}{r} = \frac{n(n-1) \cdots (n-r+1)(n-r) \cdots 2 \cdot 1}{r!(n-r) \cdots 2 \cdot 1}
\]

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The numerator
\[ n(n-1) \cdot \ldots \cdot (n-r+1)(n-r) \cdot \ldots \cdot 2 \cdot 1 \]
is
\[ n! \]
Therefore,
\[ _n C_r = \frac{n!}{r!(n-r)!} \]
This formula is read: The number of combinations of \( n \) objects taken \( r \) at a time is equal to \( n \) factorial divided by \( r \) factorial times \( n \) minus \( r \) factorial.

**EXAMPLE:** In the previous problem where 20 boys were available, how many different basketball teams could be formed?

**SOLUTION:** If the choice of which boy played center, guard, or forward is not considered, we find the number of combinations of 20 boys taken 5 at a time by writing
\[ _n C_r = \frac{n!}{r!(n-r)!} \]
where
\[ n = 20 \]
and
\[ r = 5 \]
Then, by substitution, we have
\[ _{20} C_5 = \frac{20!}{5!(20-5)!} \]
\[ = \frac{20!}{5!15!} \]
\[ = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{5!15!} \]
\[ = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \]
\[ = \frac{1 \cdot 19 \cdot 3 \cdot 17 \cdot 16}{1} \]
\[ = 15,504 \]
EXAMPLE: A man has, in his pocket, a silver dollar, a half-dollar, a quarter, a dime, a nickel, and a penny. If he reaches into his pocket and pulls out three coins, how many different sums may he have?

SOLUTION: The order is not important; therefore, the number of combinations of coins possible is

\[ c_3 = \frac{6!}{3!(6 - 3)!} \]

\[ = \frac{6!}{3!3!} \]

\[ = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!3!} \]

\[ = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} \]

\[ = \frac{5 \cdot 4}{1} \]

\[ = 20 \]

EXAMPLE: Find the value of

\[ \binom{3}{3} \]

SOLUTION: We use the formula given and find that

\[ \binom{3}{3} = \frac{3!}{3!(3 - 3)!} \]

\[ = \frac{3!}{3!0!} \]

This seems to violate the rule, division by zero is not allowed, but we define 0! to equal 1.

Then

\[ \frac{3!}{3!0!} = \frac{3!}{3!} = 1 \]

which is obvious if we list the combinations of three things taken three at a time.
PRACTICE PROBLEMS:
Find the value of problems 1 through 6 and solve problems 7, 8, and 9.

1. \( \binom{6}{2} \)
2. \( \binom{6}{4} \)
3. \( \binom{15}{5} \)
4. \( \binom{7}{7} \)
5. \( \binom{6}{3} + \frac{7 \binom{3}{3}}{\binom{13}{6}} \)
6. \( \frac{\binom{7}{3} \cdot \binom{6}{3}}{\binom{14}{4}} \)

7. We want to paint three rooms in a house, each a different color, and we may choose from seven different colors of paint. How many color combinations are possible for the three rooms?

8. If 20 boys go out for the football team, how many different teams may be formed, one at a time?

9. Two girls and their dates go to the drive-in, and each wants a different flavored ice cream cone. The drive-in has 24 flavors of ice cream. How many combinations of flavors may be chosen among the four of them if each one selects one flavor?

ANSWERS:

1. 15
2. 15
3. 3,003
4. 1
5. \( \frac{5}{156} \)
6. \( \frac{100}{143} \)
7. 35
8. 167,960
9. 10,626
PRINCIPLE OF CHOICE

The principle of choice is discussed in relation to combinations, although it is also discussed later in this chapter in relation to permutations. It is stated as follows:

*If a selection can be made in \( n_1 \) ways; and after this selection is made, a second selection can be made in \( n_2 \) ways; and after this selection is made, a third selection can be made in \( n_3 \) ways; and so forth for \( r \) selections, then the \( r \) selections can be made together in*

\[
n_1 \cdot n_2 \cdot n_3 \cdots n_r \text{ ways}
\]

*EXAMPLE:* In how many ways can a coach choose first a football team and then a basketball team from 18 boys?

*SOLUTION:* First let the coach choose a football team; that is,

\[
\begin{align*}
_{18}C_{11} &= \frac{18!}{11!(18 - 11)!} \\
&= \frac{18!}{11!7!} \\
&= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11!}{11!7!6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= 31,824
\end{align*}
\]

The coach now must choose a basketball team from the remaining seven boys; that is,

\[
\begin{align*}
_{7}C_{5} &= \frac{7!}{5!(7 - 5)!} \\
&= \frac{7!}{5!2!} \\
&= \frac{7 \cdot 6 \cdot 5!}{5!2!} \\
&= \frac{7 \cdot 6}{2} \\
&= 21
\end{align*}
\]

Then, together, the two teams can be chosen in

\[
(31,824)(21) = 668,304 \text{ ways}
\]
The same answer would be achieved if the coach chose the basketball team first and then the football team; that is,

\[
_{18}C_5 \cdot_{13}C_{11} = \frac{18!}{5!13!} \cdot \frac{13!}{11!2!}
\]

\[
= (8,568)(78)
\]

\[
= 668,304
\]

which is the same number as before.

**EXAMPLE:** A woman ordering dinner has a choice of one meat dish from four, four vegetables from seven, one salad from three, and one dessert from four. How many different menus are possible?

**SOLUTION:** The individual combinations are as follows:

- meat \( _4C_1 \)
- vegetable \( _7C_4 \)
- salad \( _3C_1 \)
- dessert \( _4C_1 \)

The values of these combinations are

\[
_{4}C_1 = \frac{4!}{1!(4-1)!} = \frac{4!}{3!} = 4
\]

and

\[
_{7}C_4 = \frac{7!}{4!(7-4)!} = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35
\]

and

\[
_{3}C_1 = \frac{3!}{1!(3-1)!} = \frac{3!}{2!} = 3
\]
Therefore, the woman has a choice of

\[ (4)(35)(3)(4) = 1,680 \]

different menus.

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**PRACTICE PROBLEMS:**

Solve the following problems:

1. A man has 12 different colored shirts and 20 different ties. How many shirt and tie combinations can he select to take on a trip if he takes 3 shirts and 5 ties?

2. A petty officer in charge of posting the watch has 12 men in his duty section. He must post 3 different fire watches and then post 4 aircraft guards on different aircraft. How many different assignments of men can he make?

3. If 10 third class and 14 second class petty officers are in a division that must furnish 2 second class and 6 third class petty officers for shore patrol, how many different shore patrol parties can be made?

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**ANSWERS:**

1. 3,410,880
2. 27,720
3. 19,110

---

**PERMUTATIONS**

Permutations are similar to combinations but extend the requirements of combinations by considering order.
Suppose we have two letters, $A$ and $B$, and wish to know how many arrangements of these letters can be made. Obviously the answer is two; that is, $AB$ and $BA$.

If we extend this to the three letters $A$, $B$, and $C$, we find the answer to be $ABC$, $ACB$, $BAC$, $BCA$, $CAB$, $CBA$.

We had three choices for the first letter; after we chose the first letter, we had only two choices for the second letter; and after the second letter, we had only one choice. This is shown in the "tree" diagram in figure 8.1. Notice that a total of six different paths lead to the ends of the "branches" of the "tree" diagram.

If the number of objects is large, the tree would become very complicated; therefore, we approach the problem in another manner, using parentheses to show the possible choices. Suppose we were to arrange six objects in as many different orders as possible. For the first choice we have six objects:

$$(6)( )( )( )( )$$

For the second choice we have only five choices:

$$(6)(5)( )( )( )$$

For the third choice we have only four choices:

$$(6)(5)(4)( )( )$$

This may be continued as follows:

$$(6)(5)(4)(3)(2)(1)$$

By applying the principle of choice, we find the total possible ways of arranging the objects to be the product of the individual choices; that is,

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and this may be written as

$$6!$$
This leads to the statement: *The number of permutations of n objects taken all together is equal to n!.*

**EXAMPLE:** How many permutations of seven different letters may be made?

**SOLUTION:** We could use a "tree" diagram, but this would become complicated. (Try it to see why.) We could use the parentheses as follows:

\[(7)(6)(5)(4)(3)(2)(1) = 5040\]

The easiest solution is to use the previous statement and write

\[7! P_7 = 7!\]

that is, the number of possible arrangements of seven objects taken seven at a time is 7!.

**NOTE:** Compare this with the number of combinations of seven objects taken seven at a time.

If we are faced with finding the number of permutations of seven objects taken three at a time, we use three parentheses as follows:

In the first position we have a choice of seven objects:

\[(7)( ) ( )\]

In the second position we have a choice of six objects:

\[(7)(6) ( )\]

In the last position we have a choice of five objects:

\[(7)(6)(5)\]

Therefore by principle of choice, the solution is

\[7 \cdot 6 \cdot 5 = 210\]

At this point we will use our knowledge of combinations to develop a formula for the number of permutations of n objects taken r at a time.
Suppose we wish to find the number of permutations of five things taken three at a time. We first determine the number of groups of three as follows:

\[ \binom{5}{3} = \frac{5!}{3!(5-3)!} \]

\[ = \frac{5!}{3!2!} \]

\[ = 10 \]

Thus, we have 10 groups of 3 objects each.

We are now asked to arrange each of these 10 groups in as many orders as possible. We know that the number of permutations of three objects taken together is 3!. We may arrange each of the 10 groups in 3! or 6 ways. The total number of possible permutations of \( \binom{5}{3} \) then is

\[ \binom{5}{3} \cdot 3! = 10 \cdot 6 \]

which can be written as

\[ \binom{5}{3} \cdot 3! = \binom{5}{3} \cdot 3! = \binom{5}{3} \cdot P_3 \]

The corresponding general form is

\[ \binom{n}{r} \cdot r! = \binom{n}{r} \cdot P_r \]

Knowing that

\[ \binom{n}{r} = \frac{n!}{r!(n-r)!} \]

then

\[ \binom{n}{r} \cdot r! = \frac{n!}{r!(n-r)!} \cdot r! \]

\[ = \frac{n!}{(n-r)!} \]

But

\[ \binom{n}{r} \cdot r! = \binom{n}{r} \cdot P_r \]

therefore,

\[ \binom{n}{r} \cdot P_r = \frac{n!}{(n-r)!} \]

This formula is read: The number of permutations of \( n \) objects taken \( r \) at a time is equal to \( n \) factorial divided by \( n \) minus \( r \) factorial.
EXAMPLE: How many permutation of six objects taken two at a time can be made?

SOLUTION: The number of permutations of six objects taken two at a time is written

\[ \binom{6}{2} = \frac{6!}{(6 - 2)!} \]

\[ = \frac{6!}{4!} \]

\[ = \frac{6 \cdot 5 \cdot 4!}{4!} \]

\[ = 6 \cdot 5 \]

\[ = 30 \]

EXAMPLE: In how many ways can eight people be arranged in a row?

SOLUTION: All eight people must be in the row; therefore, we want the number of permutations of eight people taken eight at a time, which is

\[ \binom{8}{8} = \frac{8!}{(8 - 8)!} \]

\[ = \frac{8!}{0!} \]

(remember that 0! was defined as equal to 1)

then

\[ \frac{8!}{0!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} \]

\[ = 40,320 \]

Problems dealing with combinations and permutations often require the use of both formulas to solve one problem.

EXAMPLE: Eight first class and six second class petty officers are on the board of the 56 club. In how many ways can the members elect, from the board, a president, a vice-president, a secretary, and a treasurer if the president and secretary must be first class petty officers and the vice-president and treasurer must be second class petty officers?
SOLUTION: Since two of the eight first class petty officers are to fill two different offices, we write

\[ _8P_2 = \frac{8!}{(8-2)!} \]

\[ = \frac{8!}{6!} \]

\[ = 8 \cdot 7 \]

\[ = 56 \]

Then, two of the six second class petty officers are to fill two different offices; thus, we write

\[ _6P_2 = \frac{6!}{(6-2)!} \]

\[ = \frac{6!}{4!} \]

\[ = 6 \cdot 5 \]

\[ = 30 \]

The principle of choice holds in this case; therefore, the members have

\[ 56 \cdot 30 = 1,680 \]

ways to select the required office holders.

EXAMPLE: For the preceding example, suppose we are asked the following: In how many ways can the members elect the office holders from the board if two of the office holders must be first class petty officers and two of the office holders must be second class petty officers?

SOLUTION: We have already determined how many ways eight things may be taken two at a time, how many ways six things may be taken two at a time, and how many ways they may be taken together; that is,

\[ _8P_2 = 56 \]

and

\[ _6P_2 = 30 \]

then

\[ _8P_2 \cdot _6P_2 = 1,680 \]
Our problem now is to find how many ways the members can combine the four offices two at a time. Therefore, we write

\[ _4C_2 = \frac{4!}{2!(4 - 2)!} \]

\[ = \frac{4!}{2!2!} \]

\[ = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2} \]

\[ = 6 \]

Then, in answer to the problem, we write

\[ _6P_2 \cdot _6P_2 \cdot _4C_2 = 10,080 \]

In other words, if the members have \(_4C_2\) ways of combining the four offices and then for every one of these ways, the members have \(_6P_2 \cdot _6P_2\) ways of arranging the office holders, then they have

\[ _6P_2 \cdot _6P_2 \cdot _4C_2 \]

ways of electing the petty officers.

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**PRACTICE PROBLEMS:**

Find the answers to the following:

1. \(_6P_3\)
2. \(_4P_3\)
3. \(_7P_2 \cdot _5P_2\)
4. In how many ways can six people be seated in a row?
5. Seven boys and nine girls are in a club. In how many ways can they elect four different officers designated by \(A, B, C,\) and \(D\) if
   a. \(A\) and \(B\) must be boys and \(C\) and \(D\) must be girls?
   b. two of the officers must be boys and two of the officers must be girls?
ANSWERS:

1. 120
2. 24
3. 840
4. 720
5. a. 3,024
   b. 18,144

If we were asked how many different arrangements of the letters in the word STOP can be made, we would write

\[ \frac{4!}{(4 - 4)!} \]
\[ = \frac{4!}{0!} \]
\[ = 24 \]

We would be correct since all letters are different. If some of the letters were the same, we would reason as given in the following problem.

EXAMPLE: How many different arrangements of the letters in the word ROOM can be made?

SOLUTION: We have two letters alike. If we list the possible arrangements, using subscripts to make a distinction between the O's, we have

\[
\begin{align*}
RO_1O_2M & \quad O_1O_2MR & \quad O_1MO_2R & \quad MO_1O_2R \\
RO_2O_1M & \quad O_2O_1MR & \quad O_2MO_1R & \quad MO_2O_1R \\
RO_1MO_2 & \quad O_1O_2RM & \quad O_1RMO_2 & \quad MO_1RO_2 \\
RO_2MO_1 & \quad O_2O_1RM & \quad O_2RMO_1 & \quad MO_2RO_1 \\
RMO_1O_2 & \quad O_1MRO_2 & \quad O_1RO_2M & \quad MRO_1O_2 \\
RMO_2O_1 & \quad O_2MRO_1 & \quad O_2RO_1M & \quad MRO_2O_1
\end{align*}
\]
but we cannot distinguish between the O's; RO, O₂M and RO₂O₃M would be the same arrangement without the subscript. Only half as many arrangements are possible without the use of subscripts (a total of 12 arrangements). This leads to the statement: The number of arrangements of \( n \) items, where there are \( k \) groups of like items of size \( r₁, r₂, \ldots, rₖ \), respectively, is given by

\[
\frac{n!}{r₁!r₂! \cdots rₖ!}
\]

In the previous example \( n \) was equal to 4 and two letters were alike; therefore, we would write

\[
\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 12
\]

**EXAMPLE**: How many arrangements can be made using the letters in the word **ADAPTATION**?

**SOLUTION**: We use

\[
\frac{n!}{r₁!r₂! \cdots rₖ!}
\]

where

\[
n = 10
\]

and

\[
r₁ = 2 \text{ (two T's)}
\]

and

\[
r₂ = 3 \text{ (three A's)}
\]

Then

\[
\frac{n!}{r₁!r₂! \cdots rₖ!} = \frac{10!}{2!3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2}{1} = 302,400
\]
PRACTICE PROBLEMS:

Find the number of possible arrangements of the letters in the following words:

1. WRITE
2. STRUCTURE
3. HERE
4. MILLIAMPERE
5. TENNESSEE

ANSWERS:

1. 120
2. 45,360
3. 12
4. 2,494,800
5. 3,780

Although the previous discussions have been associated with formulas, problems dealing with combinations and permutations may be analyzed and solved in a more meaningful way without complete reliance upon the formulas.

EXAMPLE: How many four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, and 7

a. without repetition?

b. with repetition?

SOLUTION: The (a) part of the question is a straightforward permutation problem, and we reason that we want
the number of permutations of six items taken four at a time. Therefore,

\[ sP_4 = \frac{6!}{(6 - 4)!} \]

\[ = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} \]

\[ = 360 \]

The (b) part of the question would become quite complicated if we tried to use the formulas; therefore, we reason as follows:

We desire a four-digit number and find we have six choices for the first position; that is, we may use any of the digits 2, 3, 4, 5, 6, or 7 in the first position:

\[ (6)\; (\;\; )\; (\;\; )\; (\;\; )\]

Since we are allowed repetition of numbers, then we have six choices for the second position. In other words, any of the digits 2, 3, 4, 5, 6, or 7 can be used in the second position:

\[ (6)(6)\; (\;\; )\; (\;\; )\]

Continuing this reasoning, we also have six choices each for the third and fourth positions:

\[ (6)(6)(6)(6)\]

Therefore, the total number of four-digit numbers formed from the digits 2, 3, 4, 5, 6, and 7 with repetition is

\[ 6 \cdot 6 \cdot 6 \cdot 6 = 1,296 \]

**EXAMPLE**: Suppose, in the previous example, we were to find how many three-digit odd numbers could be formed from the given digits without repetition.

**SOLUTION**: We would be required to start in the units column because an odd number is determined by the units column digit. Therefore, we have only three choices for the units position; that is, either 3, 5, or 7:

\[ (3)(\;\;\;)(\;\;\;\; )\]
For the ten’s position, we have only five choices, since we are not allowed repetition of numbers:

\[(3)(5)(\ )\]

Using the same reasoning of no repetition, we have only four choices for the hundred’s position:

\[(3)(5)(4)\]

Therefore, we can form

\[3 \cdot 5 \cdot 4 = 60\]

three-digit odd numbers from the digits 2, 3, 4, 5, 6, and 7 without repetition.

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**PRACTICE PROBLEMS:**

Solve the following problems:

1. Using the digits 4, 5, 6, and 7, how many two-digit numbers can be formed
   a. without repetition?
   b. with repetition?
2. Using the digits 4, 5, 6, 7, 8, and 9, how many five-digit numbers can be formed
   a. without repetition?
   b. with repetition?
3. Using the digits of problem 2, how many four-digit odd numbers can be formed without repetition?

---

**ANSWERS:**

1. a. 12
   b. 16
2. a. 720
   b. 7,776
3. 180

8-23
SUMMARY

The following are the major topics covered in this chapter:

1. Definitions:

   A combination is defined as a possible selection of a certain number of objects taken from a group without regard to order.

   A permutation is defined as a possible selection of a certain number of objects taken from a group with regard to order.

   The product of the integers $n$ through 1 is defined as $n$ factorial, and the symbol $n!$ is used to denote this.

2. Factorial: Theorem. If $n$ and $r$ are positive integers, with $n$ greater than $r$, then

   \[ n! = (n)(n - 1) \cdots (r + 2)(r + 1)r! \]

3. Combination formula:

   \[ nC_r = \frac{n!}{r!(n - r)!} \]

   for the number of combinations of $n$ objects taken $r$ at a time.

4. Principle of Choice: If a selection can be made in $n_1$ ways; and after this selection is made, a second selection can be made in $n_2$ ways; and after this selection is made, a third selection can be made in $n_3$ ways; and so forth for $r$ selections, then the sequence of $r$ selections can be made together in $n_1 \cdot n_2 \cdot n_3 \cdots n_r$ ways.

5. Permutation formula:

   \[ nP_r = \frac{n!}{(n - r)!} \]

   for the number of permutations of $n$ objects taken $r$ at a time.
6. **Arrangements**: The number of arrangements of \( n \) items, where there are \( k \) groups of like items of size \( r_1, r_2, \ldots, r_k \), respectively, is given by

\[
\frac{n!}{r_1!r_2! \cdots r_k!}
\]

7. **Repetition**: Combinations and permutation problems, with or without repetition, may be solved for using position notation instead of formulas.
ADDITIONAL PRACTICE PROBLEMS

1. Find the value of \( \frac{6!7! - 6!5!}{5!4!} \).

2. Simplify \( \frac{(n + 2)!(n - 2)! - (n + 1)!(n - 1)!}{(n)!(n - 3)!} \).

3. Find the value of \( \frac{\binom{7}{5} + \binom{7}{6}}{\binom{7}{5} - \binom{7}{6}} \).

4. On each trip, a salesman visits 4 of the 12 cities in his territory. In how many different ways can he schedule his route?

5. From six men and five women, find the number of groups of four that can be formed consisting of two men and two women.

6. Find the value of \( \frac{\binom{7}{6} + \binom{7}{5}}{\binom{7}{6} - \binom{7}{5}} \).

7. In how many ways can the 18 members of a boy scout troop elect a president, a vice-president, and a secretary, assuming that no member can hold more than one office?

8. How many different ways can 4 red, 3 blue, 4 yellow, and 2 green bulbs be arranged on a string of Christmas tree lights with 13 sockets?

9. How many car tags can be made if the first three positions are letters and the last three positions are numbers (Hint: Twenty-six letters and ten distinct single-digit numbers are possible)
   a. with repetition?
   b. without repetition?
ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 1,230
2. $3(n + 1)(n - 2)$
3. 2
4. 495
5. 150
6. 3
7. 4,896
8. 900,900
9. a. 17,576,000
   b. 11,232,000
CHAPTER 9
PROBABILITY

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Apply the basic concepts of probability.

2. Solve for probabilities of success and failure.

3. Interpret numerical and mathematical expectation.

4. Apply the concept of compound probabilities to independent, dependent, and mutually exclusive events.

5. Apply the concept of empirical events.

INTRODUCTION

The history of probability theory dates back to the 17th century and at that time was related to games of chance. In the 18th century the probability theory was known to have applications beyond the scope of games of chance. Some of the applications in which probability theory is applied are situations with outcomes such as life or death and boy or girl. Statistics and probability are currently applied to insurance, annuities, biology, and social investigations.

The treatment of probability in this chapter is limited to simple applications. These applications will be, to a large extent, based on games of chance, which lend themselves to an understanding of basic ideas of probability.

BASIC CONCEPTS

If a coin were tossed, the chance it would land heads up is just as likely as the chance it would land tails up; that is, the coin
has no more reason to land heads up than it has to land tails up. Every toss of the coin is called a trial.

We define probability as the ratio of the different number of ways a trial can succeed (or fail) to the total number of ways in which it may result. We will let \( p \) represent the probability of success and \( q \) represent the probability of failure.

One commonly misunderstood concept of probability is the effect prior trials have on a single trial. That is, after a coin has been tossed many times and every trial resulted in the coin falling heads up, will the next toss of the coin result in tails up? The answer is "not necessarily" and will be explained later in this chapter.

**PROBABILITY OF SUCCESS**

If a trial must result in any of \( n \) equally likely ways, and if \( s \) is the number of successful ways and \( f \) is the number of failing ways, the probability of success is

\[
p = \frac{s}{s + f}
\]

where

\[
s + f = n
\]

**EXAMPLE:** What is the probability that a coin will land heads up?

**SOLUTION:** There is only one way the coin can land heads up; therefore, \( s \) equals 1. There is also only one way the coin can land other than heads up; therefore, \( f \) equals 1. Since

\[s = 1\]

and

\[f = 1\]

then the probability of success is

\[
p = \frac{s}{s + f} = \frac{1}{1 + 1} = \frac{1}{2}
\]

This, then, is the ratio of successful ways in which the trial can succeed to the total number of ways the trial can result.

**EXAMPLE:** What is the probability that a die (singular of dice) will land with a 3 showing on the upper face?
**SOLUTION:** A die has a total of 6 sides. Therefore the die can land with a 3 face up 1 favorable way and 5 unfavorable ways.

Since

\[ s = 1 \]

and

\[ f = 5 \]

then

\[ p = \frac{s}{s + f} = \frac{1}{1 + 5} = \frac{1}{6} \]

**EXAMPLE:** What is the probability of drawing a black marble from a box of marbles if all 6 of the marbles in the box are white?

**SOLUTION:** There are no favorable ways of success and there are 6 total ways. Therefore,

\[ s = 0 \]

and

\[ f = 6 \]

so that

\[ p = \frac{0}{0 + 6} = \frac{0}{6} = 0 \]

**EXAMPLE:** What is the probability of drawing a black marble from a box of 6 black marbles?
SOLUTION: There are 6 successful ways and no unsuccessful ways of drawing the marble. Therefore,

\[ s = 6 \]

and

\[ f = 0 \]

so that

\[ p = \frac{6}{6 + 0} \]

\[ = \frac{6}{6} \]

\[ = 1 \]

The previous two examples arc the extremes of probabilities and intuitively demonstrate that the probability of an event ranges from 0 to 1 inclusively.

EXAMPLE: A box contains 6 hard lead pencils and 12 soft lead pencils. What is the probability of drawing a soft lead pencil from the box?

SOLUTION: We are given

\[ s = 12 \]

and

\[ f = 6 \]

therefore,

\[ p = \frac{12}{12 + 6} \]

\[ = \frac{12}{18} \]

\[ = \frac{2}{3} \]
PRACTICE PROBLEMS:

1. What is the probability of drawing an ace from a standard deck of 52 playing cards?

2. What is the probability of drawing a black ace from a standard deck of playing cards?

3. If a die is rolled, what is the probability of an odd number showing on the upper face?

4. A man has 3 nickels, 2 dimes, and 4 quarters in his pocket. If he draws a single coin from his pocket, what is the probability that
   a. he will draw a nickel?
   b. he will draw a half-dollar?
   c. he will draw a quarter?

ANSWERS:

1. \( \frac{1}{13} \)

2. \( \frac{1}{26} \)

3. \( \frac{1}{2} \)

4. a. \( \frac{1}{3} \)
   b. 0
   c. \( \frac{4}{9} \)
PROBABILITY OF FAILURE

As before, if a trial results in any of \( n \) equally likely ways, and \( s \) is the number of successful ways and \( f \) is the number of failures, the probability of failure is

\[
q = \frac{f}{s + f}
\]

or

\[
q = \frac{n - s}{n}
\]

where

\[
s + f = n
\]

or

\[
n - s = f
\]

A trial must result in either success or failure. If success is certain then \( p \) equals 1 and \( q \) equals 0. If success is impossible then \( p \) equals 0 and \( q \) equals 1. Combining both events, for either case, makes the probability of success plus the probability of failure equal to 1. If

\[
p = \frac{s}{s + f}
\]

and

\[
q = \frac{f}{s + f}
\]

then

\[
p + q = \frac{s}{s + f} + \frac{f}{s + f}
\]

\[
= 1
\]

If, in any event

\[
p + q - 1
\]

then

\[
q = 1 - p
\]
In the case of tossing a coin, the probability of success is

\[ p = \frac{s}{s + f} \]

\[ = \frac{1}{1 + 1} \]

\[ = \frac{1}{2} \]

and the probability of failure is

\[ q = 1 - p \]

\[ = 1 - \frac{1}{2} \]

\[ = \frac{1}{2} \]

**EXAMPLE**: What is the probability of not drawing a black marble from a box containing 6 white, 3 red, and 2 black marbles?

**SOLUTION**: The probability of drawing a black marble from the box is

\[ p = \frac{s}{s + f} \]

\[ = \frac{2}{2 + 9} \]

\[ = \frac{2}{11} \]

Since the probability of drawing a marble is 1, then the probability of not drawing a black marble is

\[ q = 1 - p \]

\[ = 1 - \frac{2}{11} \]

\[ = \frac{9}{11} \]
PRACTICE PROBLEMS:

Compare the following problems and answers with the preceding problems dealing with the probability of success:

1. What is the probability of not drawing an ace from a standard deck of 52 playing cards?
2. What is the probability of not drawing a black ace from a standard deck of playing cards?
3. If a die is rolled, what is the probability of an odd number not showing on the upper face?
4. A man has 3 nickels, 2 dimes, and 4 quarters in his pocket. If he draws a single coin from his pocket, what is the probability that
   a. he will not draw a nickel?
   b. he will not draw a half-dollar?
   c. he will not draw a quarter?

ANSWERS:

1. \( \frac{12}{13} \)
2. \( \frac{25}{26} \)
3. \( \frac{1}{2} \)
4. a. \( \frac{2}{3} \)
   b. 1
   c. \( \frac{5}{9} \)

EXPECTATION

*Expectation* is the average of the values you would get in conducting an experiment or trial exactly the same way many
times. In this discussion of expectation, we will consider two types. One is a numerical expectation and the other is a mathematical expectation.

**Numerical Expectation**

If you tossed a coin 50 times, you would expect the coin to fall heads (on the average) about 25 times. Your assumption is explained by the following definition of numerical expectation: *If the probability of success in one trial is p, and k is the total number of trials, then kp is the expected number of successes in the k trials.*

In the above example of tossing the coin 50 times, the probability of heads (successes) is

\[ E_n = kp \]

where

\[ E_n = \text{expected number} \]
\[ k = \text{number of tosses} \]
\[ p = \text{probability of heads (successes)} \]

Substituting values in the equation, we find that

\[ E_n = 50 \left( \frac{1}{2} \right) \]
\[ = 25 \]

**EXAMPLE**: A die is rolled by a player. What is the expectation of rolling a 6 in 30 trials?

**SOLUTION**: The probability of rolling a 6 in 1 trial is

\[ p = \frac{1}{6} \]

and the number of rolls is

\[ k = 30 \]

therefore,

\[ E_n = kp \]
\[ = 30 \left( \frac{1}{6} \right) \]
\[ = 5 \]
In words, the player would expect (on the average) to roll a 6 five times in 30 rolls.

**Mathematical Expectation**

We will define *mathematical expectation* as follows: *If, in the event of a successful result, amount a, is to be received and the probability of success of that event is p, then ap is the mathematical expectation.*

If you were to buy 1 of 500 raffle tickets for a video recorder worth $325.00, what would be your mathematical expectation?

In this case, the product of the amount you stand to win and the probability of winning is

\[ E_m = ap \]

where

\[ a = \text{amount you stand to win} \]

\[ p = \text{probability of success} \]

and

\[ E_m = \text{expected amount} \]

Then, by substitution

\[ E_m = ap \]

\[ = 325.00 \left( \frac{1}{500} \right) \]

\[ = 0.65 \]

Thus, you would not want to pay more than 65 cents for the ticket, unless, of course the raffle were for a worthy cause.

**EXAMPLE:** To entice the public to invest in their development, Sunshine Condominiums has offered a prize of $2,000 to 1 randomly selected family out of the first 1,000 families that participate in the condominium’s tour.

1. What would be each family’s mathematical expectation?

2. Would it be worthwhile for the Jones family to spend $3.00 in gasoline to drive to Sunshine Condominiums to take the tour?
SOLUTION:

1. \( E_m = ap = 2,000 \left( \frac{1}{1,000} \right) = 2.00 \)

2. No; since $3.00 is $1.00 over their expectation of $2.00, it would not be worthwhile for the Jones family to take the tour.

PRACTICE PROBLEMS:

1. A box contains 7 slips of paper, each numbered differently. A girl makes a total of 50 draws, returning the drawn slip after each draw.
   a. What is the probability of drawing a selected numbered slip in 1 drawing?
   b. How many times would the girl expect to draw the single selected numbered slip in the 50 draws?

2. In a winner-take-all tournament among four professional tennis players, the prize money is $500,000. Joe Conners, one of the tennis players, figures his probability of winning is 0.20.
   a. What is his mathematical expectation?
   b. Would he be better off if he made a secret agreement with the other tennis players to divide the prize money evenly regardless of who wins?

ANSWERS:

1. a. \( \frac{1}{7} \)
   b. \( 7 \frac{1}{7} \)

2. a. $100,000
   b. Yes; he would be better off, since he would make $125,000, which is greater than his expectation of $100,000.
COMPOUND PROBABILITIES

The probabilities to this point have been single events. In the discussion on compound probabilities, events that may affect others will be covered. The word may is used because independent events are included with dependent events and mutually exclusive events.

INDEPENDENT EVENTS

Two or more events are independent if the occurrence or nonoccurrence of one of the events has no affect on the probability of occurrence of any of the others.

When two coins are tossed at the same time or one after the other, whether one falls heads or tails has no affect on the way the second coin falls. Suppose we call the coins $A$ and $B$. The coins may fall in the following four ways:

1. $A$ and $B$ may fall heads.
2. $A$ and $B$ may fall tails.
3. $A$ may fall heads and $B$ may fall tails.
4. $A$ may fall tails and $B$ may fall heads.

The probability of any one way for the coins to fall is calculated as follows:

\[ s = 1 \]

and

\[ n = 4 \]

therefore,

\[ P = \frac{1}{4} \]

This probability may be determined by considering the product of the separate probabilities; that is,

the probability that $A$ will fall heads is $\frac{1}{2}$

the probability that $B$ will fall heads is $\frac{1}{2}$

and the probability that both will fall heads is

\[ \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \]
In other words, when two events are independent, the probability that one and then the other will occur is the product of their separate probabilities.

**EXAMPLE:** A box contains 3 red marbles and 7 green marbles. If a marble is drawn, then replaced, and another marble is drawn, what is the probability that both marbles are red?

**SOLUTION:** Two solutions are offered. First, by the principle of choice, 2 marbles can be selected in $10 \cdot 10$ ways. The red marble may be selected on the first draw in three ways and on the second draw in three ways; and by the principle of choice, a red marble may be drawn on both trials in $3 \cdot 3$ ways. Then the required probability is

$$p = \frac{9}{100}$$

The second solution, using the product of independent events, follows: The probability of drawing a red marble on the first draw is $\frac{3}{10}$, and the probability of drawing a red marble on the second draw is $\frac{3}{10}$. Therefore, the probability of drawing a red marble on both draws is the product of the separate probabilities or

$$p = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100}$$

**PRACTICE PROBLEMS:**

1. If a die is tossed twice, what is the probability of rolling a 2 followed by a 3?

2. A box contains 2 white, 3 red, and 4 blue marbles. If after each selection the marble is replaced, what is the probability of drawing, in order

   a. a white then a blue marble?

   b. a blue then a red marble?

   c. a white, a red, then a blue marble?
ANSWERS:

1. \( \frac{1}{36} \)

2. a. \( \frac{8}{81} \)
   
b. \( \frac{4}{27} \)
   
c. \( \frac{8}{243} \)

DEPENDENT EVENTS

In some cases one event is dependent on another; that is, two or more events are said to be dependent if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others.

Consider that two or more events are dependent. If \( p_1 \) is the probability of a first event; \( p_2 \) the probability that after the first happens, the second will occur; \( p_3 \) the probability that after the first and second have happened, the third will occur; etc., then the probability that all events will happen in the given order is the product \( p_1 \cdot p_2 \cdot p_3 \cdot \ldots \).

EXAMPLE: A box contains 3 white marbles and 4 black marbles. What is the probability of drawing 2 black marbles and 1 white marble in succession without replacement?

SOLUTION: On the first draw the probability of drawing a black marble is

\[ p_1 = \frac{4}{7} \]

On the second draw the probability of drawing a black marble is

\[ p_2 = \frac{3}{6} = \frac{1}{2} \]
On the third draw the probability of drawing a white marble is

\[ p_3 = \frac{3}{5} \]

Therefore, the probability of drawing 2 black marbles and 1 white marble is

\[ p = p_1 \cdot p_2 \cdot p_3 \]
\[ = \frac{4}{7} \cdot \frac{1}{2} \cdot \frac{3}{5} \]
\[ = \frac{6}{35} \]

**EXAMPLE:** Slips numbered 1 through 9 are placed in a box. If 2 slips are drawn, without replacement, what is the probability that

1. both are odd?
2. both are even?

**SOLUTION:**

1. The probability that the first is odd is

\[ p_1 = \frac{5}{9} \]

and the probability that the second is odd is

\[ p_2 = \frac{4}{8} \]

Therefore, the probability that both are odd is

\[ p = p_1 \cdot p_2 \]
\[ = \frac{5}{9} \cdot \frac{4}{8} \]
\[ = \frac{5}{18} \]
2. The probability that the first is even is

\[ p_1 = \frac{4}{9} \]

and the probability that the second is even is

\[ p_2 = \frac{3}{8} \]

Therefore, the probability that both are even is

\[ p = p_1 \cdot p_2 \]
\[ = \frac{4}{9} \cdot \frac{3}{8} \]
\[ = \frac{1}{6} \]

A second method of solution involves the use of combinations.

1. A total of 9 slips are taken 2 at a time and 5 odd slips are taken 2 at a time; therefore,

\[ p = \frac{5}{9} \cdot \frac{3}{8} \]
\[ = \frac{5}{18} \]

2. A total of 5 \( \binom{9}{2} \) choices and 4 even slips are taken 2 at a time; therefore,

\[ p = \frac{4}{9} \cdot \frac{5}{8} \]
\[ = \frac{1}{6} \]

PRACTICE PROBLEMS:

In the following problems assume that no replacement is made after each selection:

1. A box contains 5 white and 6 red marbles. What is the probability of successfully drawing, in order, a red marble and then a white marble?
2. A bag contains 3 red, 2 white, and 6 blue marbles. What is the probability of drawing, in order, 2 red, 1 blue, and 2 white marbles?

3. Fifteen airmen are in the line crew. They must take care of the coffee mess and line shack cleanup. They put slips numbered 1 through 15 in a hat and decide that anyone who draws a number divisible by 5 will be assigned the coffee mess and anyone who draws a number divisible by 4 will be assigned cleanup. The first person draws a 4, the second a 3, and the third an 11. What is the probability that the fourth person to draw will be assigned

   a. the coffee mess?

   b. the cleanup?

ANSWERS:

1. \( \frac{3}{11} \)

2. \( \frac{1}{770} \)

3. a. \( \frac{1}{4} \)

   b. \( \frac{1}{6} \)

MUTUALLY EXCLUSIVE EVENTS

Two or more events are called mutually exclusive if the occurrence of any one of them precludes the occurrence of any of the others. The probability of occurrence of two or more mutually exclusive events is the sum of the probabilities of the individual events.

Sometimes when one event has occurred, the probability of another event is excluded (referring to the same given occasion or trial).
For example, throwing a die once can yield a 5 or 6, but not both, in the same toss. The probability that either a 5 or 6 occurs is the sum of their individual probabilities.

\[ p = p_1 + p_2 \]

\[ = \frac{1}{6} + \frac{1}{6} \]

\[ = \frac{1}{3} \]

**EXAMPLE:** From a bag containing 5 white balls, 2 black balls, and 11 red balls, 1 ball is drawn. What is the probability that it is either black or red?

**SOLUTION:** The draw can be made in 18 ways. The choices are 2 black balls and 11 red balls, which are favorable, or a total of 13 favorable choices. Then, the probability of success is

\[ p = \frac{13}{18} \]

Since drawing a red ball excludes the drawing of a black ball, and vice versa, the two events are mutually exclusive; so the probability of drawing a black ball is

\[ p_1 = \frac{2}{18} \]

and the probability of drawing a red ball is

\[ p_2 = \frac{11}{18} \]

Therefore, the probability of success is

\[ p = p_1 + p_2 \]

\[ = \frac{2}{18} + \frac{11}{18} = \frac{13}{18} \]

**EXAMPLE:** What is the probability of drawing either a king, a queen, or a jack from a deck of playing cards?

9-18
**SOLUTION:** The individual probabilities are

\[
\text{king} = \frac{4}{52} \\
\text{queen} = \frac{4}{52} \\
\text{jack} = \frac{4}{52}
\]

Therefore, the probability of success is

\[
p = \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52} = \frac{3}{13}
\]

**EXAMPLE:** What is the probability of rolling a die twice and having a 5 and then a 3 show or having a 2 and then a 4 show?

**SOLUTION:** The probability of having a 5 and then a 3 show is

\[
p_1 = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}
\]

and the probability of having a 2 and then a 4 show is

\[
p_2 = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}
\]

Then, the probability of either \(p_1\) or \(p_2\) is

\[
p = p_1 + p_2 = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}
\]
PRACTICE PROBLEMS:

1. When tossing a coin, you have what probability of getting either a head or a tail?

2. A bag contains 12 blue, 3 red, and 4 white marbles. What is the probability of drawing
   a. in 1 draw, either a red or a white marble?
   b. in 1 draw, either a red, white, or blue marble?
   c. in 2 draws, either a red marble followed by a blue marble or a red marble followed by a red marble?

3. What is the probability of getting a total of at least 10 points in rolling two dice? (HINT: You want either a total of 10, 11, or 12.)

ANSWERS:

1. 1

2. a. \( \frac{7}{19} \)
   b. 1
   c. \( \frac{7}{57} \)

3. \( \frac{1}{6} \)

EMPIRICAL PROBABILITIES

Among the most important applications of probability are those situations where we cannot list all possible outcomes. To this point, we have considered problems in which the probabilities could be obtained from situations of equally likely results.
Because some problems are so complicated for analysis, we can only estimate probabilities from experience and observa-
tion. This is empirical probability.

In modern industry probability now plays an important role in many activities. Quality control and reliability of a
manufactured article have become extremely important considerations in which probability is used.

Experience has shown that empirical probabilities, if carefully determined on the basis of adequate statistical samples,
can be applied to large groups with the result that probability and relative frequency are approximately equal. By ade-
quate samples we mean a large enough sample so that accidental runs of "luck," both good and bad, cancel each other.
With enough trials, predicted results and actual results agree quite closely. On the other hand, applying a probability ratio
to a single individual event is virtually meaningless.

We define relative frequency of success as follows: After $N$
trials of an event have been made, of which $S$ trials are successes,
the relative frequency of success is

\[ P = \frac{S}{N} \]

For example, table 9-1 shows a small number of weather forecasts from April 1st to April 10th. The actual weather on the
dates is also given.

Observe that the forecasts on April 1, 3, 4, 6, 7, 8, and 10 were correct. We have observed 10 outcomes. The event of a cor-
correct forecast has occurred 7 times. Based on this information we might say that the probability for future forecasts being true is
7/10. This number is the best estimate we can make from the given
information. In this case, since we have observed such a small number of outcomes, we would be incorrect to say that the
estimate of $P$ is dependable. A great many more cases should be
used if we expect to make a good estimate of the probability that
a weather forecast will be accurate. A great many factors affect
the accuracy of a weather forecast. This example merely indicates
something about how successful a particular weather office has
been in making weather forecasts.

<table>
<thead>
<tr>
<th></th>
<th>Date</th>
<th>Forecast</th>
<th>Actual weather</th>
<th>Did the actual forecasted event occur?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Rain</td>
<td>Rain</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Light showers</td>
<td>Sunny</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Cloudy</td>
<td>Cloudy</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Clear</td>
<td>Clear</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Scattered showers</td>
<td>Warm and sunny</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Scattered showers</td>
<td>Scattered showers</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>Windy and cloudy</td>
<td>Windy and cloudy</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Thundershowers</td>
<td>Thundershowers</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Clear</td>
<td>Cloudy and rain</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Clear</td>
<td>Clear</td>
<td>Yes</td>
<td></td>
</tr>
</tbody>
</table>
Another example may be drawn from industry. Many thousands of articles of a certain type are manufactured. The company selects 100 of these articles at random and subjects them to very careful tests. In these tests 98 of the articles are found to meet all measurement requirements and perform satisfactorily. This suggests that 98/100 is a measure of the reliability of the article.

One might expect that about 98% of all of the articles manufactured by this process will be satisfactory. The probability (measure of chance) that one of these articles will be satisfactory might be said to be 0.98.

This second example of empirical probability is different from the first example in one very important respect. In the first example we could list all of the possibilities, and in the second example we could not do so. The selection of a sample and its size is a problem of statistics.

Considered from another point of view, statistical probability can be regarded as relative frequency.

**EXAMPLE:** In a dart game, a player hit the bull’s eye 3 times out of 25 trials. What is the statistical probability that he will hit the bull’s eye on the next throw?

**SOLUTION:**

\[ N = 25 \]

and

\[ S = 3 \]

hence

\[ P = \frac{3}{25} \]

**EXAMPLE:** Using table 9-2, what is the probability that a person 20 years old will live to be 50 years old?

**SOLUTION:** Of 95,148 persons at age 20, 81,090 survived to age 50. Hence

\[ P = \frac{81,090}{95,148} \]

= 0.852 (rounded)

**EXAMPLE:** How many times would a die be expected to land with a 5 or 6 showing in 20 trials?
SOLUTION: The probability of a 5 or 6 showing is

\[ p = \frac{1}{3} \]

The relative frequency is approximately equal to the probability

\[ P \approx p \]

Therefore, since

\[ P = \frac{S}{N} \]

where

\[ P = \frac{1}{3} \]
\[ N = 20 \]
\[ S = ? \]

then rearranging and substituting, we find that

\[ S = NP \]
\[ = 20 \left( \frac{1}{3} \right) \]
\[ = \frac{20}{3} \]
\[ = 6.67 \text{ (rounded)} \]

This says that the expected number of times a die would land with a 5 or 6 showing in 20 trials is 6.67; that is, on the average a die will land with a 5 or 6 showing 6.67 times per 20 trials.

PRACTICE PROBLEMS:

1. A construction crew consists of 6 electricians and 38 other workers. How many electricians would you expect to choose if you choose 1 person each day of a workweek for your
helper? (Sunday will not be considered part of the workweek.)

2. How many times would a tossed die be expected to turn up a 3 or less in 30 tosses?

3. Using table 9-2, find the probability that a person whose age is 30 will live to age 60.

**ANSWERS:**

1. 0.82

2. 15

3. 0.733
SUMMARY

The following are the major topics covered in this chapter:

1. **Probability**: *Probability* is the ratio of the different number of ways a trial can succeed (or fail) to the total number of ways in which it may result.

   \[ p = \frac{s}{s + f} \]

   and the probability of failure is

   \[ q = \frac{f}{s + f} \text{ or } q = \frac{n - s}{n} \]

   where \( s + f = n \) or \( n - s = f \)

2. **Probabilities of success and failure**: If a trial must result in any of \( n \) equally likely ways, and if \( s \) is the number of successful ways and \( f \) is the number of failing ways, the probability of success is

3. **Expectation**: *Expectation* is the average of the values you would get in conducting an experiment or trial exactly the same way many times.

4. **Numerical expectation**: If the probability of success in one trial is \( p \), and \( k \) is the total number of trials, then \( kp \) is the expected number of successes in the \( k \) trials or

   \[ E_n = kp \]

5. **Mathematical expectation**: If, in the event of a successful result, amount \( a \) is to be received, and \( p \) is the probability of success of that event, then \( ap \) is the mathematical expectation or

   \[ E_m = ap \]

6. **Independent events**: Two or more events are *independent* if the occurrence or nonoccurrence of one of the events has no effect on the probability of occurrence of any of the others.

7. **Dependent events**: Two or more events are *dependent* if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others.
8. **Mutually exclusive events:** Two or more events are called *mutually exclusive* if the occurrence of any one of them precludes the occurrence of any of the others.

9. **Empirical probability:** *Empirical probability* is an estimated probability from experience and observation.

10. **Relative frequency of success:** After $N$ trials of an event have been made, of which $S$ trials are successes, the relative frequency of success is

$$P = \frac{S}{N}$$
ADDITIONAL PRACTICE PROBLEMS

1. A box contains 5 red marbles, 6 blue marbles, and 7 green marbles. If 1 marble is to be drawn, what is the probability that it is
   
   a. green?
   
   b. red?
   
   c. yellow?

2. A box contains 5 red marbles, 6 blue marbles, and 7 green marbles. If 1 marble is to be drawn, what is the probability that it is not
   
   a. green?
   
   b. red?
   
   c. yellow?

3. A child is to pick a letter of the alphabet from a box.
   
   a. What is the child’s probability of picking a vowel in 1 draw? (Y will not be considered as a vowel.)
   
   b. What is the child’s numerical expectation of picking a vowel in 20 draws? (The letter will be replaced after each draw.)

4. A concert promoter agrees to pay a band $5,600 in case the concert has to be cancelled because of rain. The promoter’s actuary figures expected loss for this risk to be $717. What probability is assigned to the possibility that the concert will have to be cancelled because of rain?

5. A basket contains 3 apples, 5 pears, and 7 oranges. If after each selection the fruit is replaced, what is the probability of drawing, in order,
   
   a. an orange, then a pear?
   
   b. 2 apples?
   
   c. an apple, an orange, then a pear?

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6. A car has 8 spark plugs, of which 3 are defective. Find the probability of locating all 3 defective spark plugs in 3 selections, without replacement.

7. In a convention of 120 politicians, 52 are Democrats and 33 are Republicans. Find the probability that a politician selected is a Democrat or a Republican.

8. Routine medical examinations are given to 44 smokers and 62 nonsmokers. If one of the subjects is selected for more detailed tests, what is the probability that the selected subject smokes?

9. In a classroom of 33 girls and 22 boys, how many girls would you expect to choose in 12 trials?
ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. a. 7/18
   b. 5/18
   c. 0
2. a. 11/18
   b. 13/18
   c. 1
3. a. 5/26
   b. 3.85
4. 0.128
5. a. 7/45
   b. 1/25
   c. 7/225
6. 1/56
7. 17/24
8. 22/53
9. 7.2