Lecture 7

Fourier Series & Spectrum

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READING ASSIGNMENTS

• This Lecture:
  – Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6
  • Replaces pp. 62-66 in Ch 3 in DSP First
  • Notation: \( a_k \) for Fourier Series

• Other Reading:
  – Next Lecture: Sampling

LECTURE OBJECTIVES

• ANALYSIS via Fourier Series
  – For PERIODIC signals: \( x(t+T_0) = x(t) \)
    \[ a_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j(2\pi k / T_0)t} \, dt \]

• SPECTRUM from Fourier Series
  – \( a_k \) is Complex Amplitude for \( k \)-th Harmonic

SPECTRUM DIAGRAM

• Recall Complex Amplitude vs. Freq
  \[ x(t) = a_0 + \sum_{k=1}^{10} \left( a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right) \]
Harmonic Signal

\[ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t} \]

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

\[ 2\pi f_0 = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0} \]

Example

\[ x(t) = \sin^3(3\pi t) \]

\[ x(t) = \left( \frac{1}{8} e^{j\pi/6} + \frac{-3j}{8} e^{j\pi/2} + \frac{3j}{8} e^{j\pi} + \frac{-j}{8} e^{-j\pi/2} \right) e^{-j\pi t} \]

Example

\[ x(t) = \sin^3(3\pi t) \]

\[ x(t) = \left( \frac{1}{8} e^{j\pi/6} + \frac{-3j}{8} e^{j\pi/2} + \frac{3j}{8} e^{j\pi} + \frac{-j}{8} e^{-j\pi/2} \right) e^{-j\pi t} \]

In this case, analysis just requires picking off the coefficients.

\[ a_k = \begin{cases} \frac{1}{2} & k = -3 \\ \frac{1}{2} & k = -1 \\ \frac{1}{2} & k = 1 \\ \frac{1}{2} & k = 3 \end{cases} \]

Frequency in Hz

FS: Rectified Sine Wave \( \{a_k\} \)

\[ a_k = \frac{1}{T_0} \int_{0}^{T_0} x(t) e^{-j\omega_0 k t} dt \quad (k \neq \pm 1) \]

Half-Wave Rectified Sine

FS: Rectified Sine Wave \( \{a_k\} \)

\[ a_k = \begin{cases} \frac{1}{2} e^{j(2\pi/3)k} & k \neq \pm 1, \pm 2, \pm 3, \pm 4 \\ \frac{1}{2} \int_{0}^{T_0/2} e^{j(2\pi/3)k} e^{j2\pi f_k t} dt & k = \pm 1, \pm 2, \pm 3, \pm 4 \end{cases} \]

For \( k \) odd:

\[ a_k = \frac{1}{2} \int_{0}^{T_0/2} \left( e^{j(2\pi/3)k} - 1 \right) e^{-j\pi k} e^{-j(2\pi/3)(k/2)} \frac{1}{2} \frac{k}{k+1} \left( k \frac{-1}{k+1} \right) - 1 \]

For \( k \) even:

\[ a_k = \frac{1}{2} \int_{0}^{T_0/2} \left( e^{j(2\pi/3)k} - 1 \right) e^{-j\pi k} e^{-j(2\pi/3)(k/2)} \frac{1}{2} \frac{k}{k+1} \left( k \frac{-1}{k+1} \right) - 1 \]

For \( k = \pm 1, \pm 3 \):

\[ a_k = \frac{1}{2} \int_{0}^{T_0/2} \left( e^{j(2\pi/3)k} - 1 \right) e^{-j\pi k} e^{-j(2\pi/3)(k/2)} \frac{1}{2} \frac{k}{k+1} \left( k \frac{-1}{k+1} \right) - 1 \]
**SQUARE WAVE EXAMPLE**

\[
x(t) = \begin{cases} 1 & 0 \leq t < \frac{T_0}{2} \\ 0 & \frac{T_0}{2} \leq t < T_0 \end{cases}
\]

for \( T_0 = 0.04 \) sec.

\[ x(t) \]

---

**FS for a SQUARE WAVE  \{a_k\}**

\[
a_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)
\]

\[
a_k = \frac{1}{T_0} \int_0^{T_0} e^{-j(2\pi/0.04)kt} dt = \frac{1}{j2\pi k} \left( e^{-j(2\pi/0.04)kT_0} - 1 \right) = \frac{1}{j2\pi k} \left( -1 + e^{-j(2\pi/0.04)kT_0} \right)
\]

**DC Coefficient:  \(a_0\)**

\[
a_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt} dt \quad (k = 0)
\]

\[
a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} \text{(Area)}
\]

\[
a_0 = \frac{1}{0.04} \int_0^{0.02} dt = \frac{1}{0.04} (0.02 - 0) = \frac{1}{2}
\]

**Fourier Coefficients  \(a_k\)**

- \(a_k\) is a function of \(k\)
  - Complex Amplitude for \(k\)-th Harmonic
  - This one doesn’t depend on the period, \(T_0\)

\[
a_k = 1 - \left( \frac{1}{j2\pi k} \right) = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}
\]

**Spectrum from Fourier Series**

\[
a_k = 2\pi f_0(0.04) = 2\pi(25)
\]

**Fourier Series Synthesis**

- HOW do you APPROXIMATE \(x(t)\)?

\[
a_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt} dt
\]

- Use FINITE number of coefficients

\[
x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}
\]

\[ a_k = a_k^* \quad \text{when } x(t) \text{ is real} \]
Fourier Series Synthesis

Fourier Series Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi} \cos(2\pi(25)t - \frac{\pi}{2}) + \frac{2}{3\pi} \cos(2\pi(75)t - \frac{\pi}{2})$$

Fourier Synthesis

$$s_N(t) = \frac{1}{2} + \frac{2}{\pi} \sin(\omega_0 t) + \frac{2}{3\pi} \sin(3\omega_0 t) + \ldots$$

Gibbs’ Phenomenon

- Convergence at DISCONTINUITY of $$x(t)$$
  - There is always an overshoot
  - 9% for the Square Wave case

Fourier Series Demos

- Fourier Series Java Applet
  - Greg Slabaugh
    - Interactive
    - http://users.ece.gatech.edu/mcclella/2025/Fourier_Slabaugh/fourier.html

- MATLAB GUI: fseriesdemo
  - http://users.ece.gatech.edu/mcclella/matlabGUIs/index.html
fseriesdemo GUI

Fourier Series Java Applet

Harmonic Signal (3 Freqs)