Short-Time Fourier Transform: Two Fundamental Properties and an Optimal Implementation

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Abstract—Shift and rotation invariance properties of linear time-frequency representations are investigated. It is shown that among all linear time-frequency representations, only the short-time Fourier transform (STFT) family with the Hermite–Gaussian kernels satisfies both the shift invariance and rotation invariance properties that are satisfied by the Wigner distribution (WD). By extending the time-bandwidth product (TBP) concept to fractional Fourier domains, a generalized time-bandwidth product (GTBP) is defined. For mono-component signals, it is shown that GTBP provides a rotation independent measure of compactness. Similar to the TBP optimal STFT, the GTBP optimal STFT that causes the least amount of increase in the GTBP of the signal is obtained. Finally, a linear canonical decomposition of the obtained GTBP optimal STFT analysis is presented to identify its relation to the rotationally invariant STFT.

Index Terms—Fractional Fourier transform, generalized time-bandwidth product, linear time-frequency representations, rotation invariance, short-time Fourier transform.

I. INTRODUCTION

Research on time-frequency domain characterization of signals has been focused on the variants of short-time Fourier transform (STFT) [1]–[7] and Wigner distribution (WD) [8], [9]. The absence of undesirable cross terms [1], [10] and computational simplicity [11]–[13] are the major factors in the wide-spread use of the STFT in practice. With the advance of faster processors, the efficiency of the STFT techniques have become less important. However, the ability of representing time-frequency content of signals free of cross terms is still the major advantage of the STFT techniques over the WD-related quadratic time-frequency distributions.

Among its many important properties, the STFT has a fundamental property that simplifies the interpretation of the resultant distribution: magnitude-wise shift invariance in both time and frequency. In this paper, we first prove that the STFT is the only linear distribution that has the magnitude-wise shift invariance property in both time and frequency. Then, we investigate time-frequency domain rotation property within the general class of linear distributions. This lesser known property, which is satisfied by the WD, is defined as follows: A time-frequency distribution satisfies the rotation property if the distribution of an arbitrary signal and the distribution of its \( \alpha \)-th order fractional Fourier transformation are \( \alpha \pi/2 \) rad rotated versions of each other [14], [15]. We start our investigation on linear time-frequency distributions by showing that STFT satisfies the rotation property only if the STFT kernel is a Hermite-Gaussian function. Thus, we reach the conclusion that the linear time-frequency distributions, which satisfy both the rotation property and the magnitude-wise time and frequency shift property, are the STFT with Hermite-Gaussian kernels.

The choice of the STFT kernel determines the time-frequency signal localization properties of the distribution. Among the Hermite-Gaussian function family, since it has the minimum time-bandwidth product (TBP), the Gaussian function is the most commonly used kernel function. However, STFT with the Gaussian kernel still suffers from the problem of limited resolution. To overcome the inherent tradeoff between the time and the frequency localization of the STFT, several alternatives have been investigated in the literature. In [5], using two kernel functions of different supports, a wideband and a narrowband spectrogram are obtained. In order to preserve the localization characteristics of both, a combined spectrogram is formed by computing the geometric mean of the corresponding STFT magnitudes, whereas in [6], the STFT is evaluated by using a kernel function with an adaptive width in order to analyze the transient response of radar targets. In [16], a kernel matching algorithm is developed by locally adapting the Gaussian kernel functions to the analyzed signal. Although these investigations provide significant improvements in the time-frequency localization of signal components, in the presence of chirp-like signals, they still provide descriptions whose localization properties depend on the chirp rate of the components. Recently, [7] introduced an improved instantaneous frequency estimation technique using an adaptive STFT where the kernel functions are chosen from a set of functions through adaptation rules and computation of the STFT with varying kernel functions at each time instance. In addition, apart from the analysis of deterministic signals, there have been studies where time-varying spectra of random processes are investigated [17].

In this paper, we characterize the time-frequency domain localization by STFT and investigate the effect of the STFT kernel on the obtained time-frequency representation of signals. We introduce the generalized time-bandwidth product (GTBP) definition to provide a rotation-invariant measure of signal support in the time-frequency domain. Then, we obtain the optimal STFT kernel that provides the most compact representation considering the GTBP of a signal component. The proposed time-frequency analysis is shown to be equivalent to an ordinary STFT analysis conducted in a scaled fractional Fourier transform domain. The obtained GTBP optimal STFT representation yields...