EXAMPLE 2: $V_g = 100 \, \text{V}$, $Z_g = 50 \, \Omega$, $f = 10^8 \, \text{Hz}$, $R_0 = 50 \, \Omega$

$Z_L = 25 + j \, 25 \, \Omega$ ve $l = 3.6 \, \text{m}$ are given. Find out that

a) $V(z)$ b) $V_i$ c) $V_L$ d) VSWR e) $P_L$ = ?

**Solution:** As we have $Z_o = Z_G$ so the source is matched to the line in this case we have $\Gamma_G = 0$.

$$V(z) = V_0^+ e^{-j \beta z} \left(1 + \Gamma(z)\right), \quad \Gamma(z) = \Gamma_L e^{-j \frac{2}{L} \beta d}$$

$$V(z) = \frac{V_G}{Z_0 + Z_G} \cdot Z_G \cdot e^{-j \beta z} \left(1 + \Gamma_L e^{-j \frac{2}{L} \beta d}\right)$$

where

$$V_0^+ = \frac{V_G}{Z_0 + Z_G} \cdot Z_G, \quad d = 1 - z$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25 + j25 - 50}{25 + j25 + 50} = \frac{-25 + j25}{75 + j25}$$

$$= \frac{-35}{79} e^{-j135^\circ}$$

$$\Gamma_L = 0.44 e^{j116.57^\circ}$$
\[ \beta = \frac{2\pi}{3}, \quad V(z) = 50 \ e^{-j\frac{2\pi}{3}z} \left(1 + 0.44 \ e^{-j\left(\frac{4\pi}{3}z - 0.128\pi\right)}\right) \]

b) \[ V_i = V(0) = 50 \left(1 + 0.44 \ e^{j(-0.128\pi)} \right) \]

c) \[ V_L = V(z=3.6m) \]

d) \[ \text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.44}{1 - 0.44} \]

e) \[ P = \frac{1}{2} \left|\frac{V_L}{Z_L}\right|^2 \quad R \quad P = 0.119 \text{ W} \]
SMITH CHART

Transmission-line calculations such as the determination of input impedance, reflection coefficient and load impedance often involve tedious manipulations of complex numbers. This tedium can be alleviated by using a graphical method of solution. The best known and most widely used graphical chart is the Smith chart devised by P.H. Smith in 1939. Smith chart is a graphical plot of normalized resistance and reactance functions in the reflection-coefficient plane.

In order to understand how the Smith chart for a lossless transmission line is constructed, let us examine the voltage reflection coefficient of the load impedance:

\[
\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = |\Gamma| e^{j\varphi_r} \quad (133)
\]

Let the load impedance be normalized with respect to the characteristic impedance of the line.

\[
z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx \quad (134)
\]

where \(r\) and \(x\) are the normalized resistance and normalized reactance respectively. Equation (133) can be rewritten as

\[
\Gamma = \Gamma_r + j\Gamma_i = \frac{z_L - 1}{z_L + 1} \quad (135)
\]

where \(\Gamma_r\) and \(\Gamma_i\) are the real and imaginary parts of the voltage reflection coefficient respectively. The inverse relation of equation (135) is

\[
z_L = \frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + |\Gamma| e^{j\varphi_r}}{1 - |\Gamma| e^{j\varphi_r}} \quad (136)
\]

or
\[ r + jx = \frac{(1 + \Gamma_r) + j\Gamma_i}{(1 - \Gamma_r) - j\Gamma_i} \]  

(137)

Multiplying both the numerator and the denominator of equation (137) by the complex conjugate of the denominator, and separating the real and imaginary parts of both sides, we obtain

\[ r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \]  

(138)

and

\[ x = \frac{2\Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \]  

(139)

If equation (138) is plotted in the plane for a given value of \( r \), the resulting graph is the locus for this value of \( r \). The locus can be recognized when the equation is rearranged as

\[ \left( \Gamma_r - \frac{r}{1+r} \right)^2 + \Gamma_i^2 = \left( \frac{1}{1+r} \right)^2 \]  

(140)

It is the equation for a circle having a radius of \( 1/(1+r) \) and centered at \((r/(1+r),0)\). Different values of \( r \) yield circles of different positions in the reflection coefficient plane. A family of these circles are shown in figure 1. Since only that part of graph lying within the unit circle on the plane is meaningful; everything the outside can be disregarded.
Several salient properties of the $r$-circles are noted as follows:

1. The centers of all $r$-circles lie on the $\Gamma_r$–axis.

2. The $r = 0$ circle, having a unity radius and centered at the origin, is the largest.

3. The $r$-circles become progressively smaller as $r$ increases from 0 toward $\infty$, ending at the $(\Gamma_r = 1, \Gamma_i = 0)$ point.

4. All $r$-circles pass through the $(\Gamma_r = 1, \Gamma_i = 0)$ point.

Similarly, (139) may be rearranged as

\[
(\Gamma_r - 1)^2 + (\Gamma_i - 1/x)^2 = (1/x)^2
\]

This is the equation for a circles having radius $1/|x|$ and centered at
\( \Gamma_r = 1 \) and \( \Gamma_i = 1/x \).

Different values of \( x \) yield circles of different radii with centers at different position on the \( \Gamma_r = 1 \) line. A family of the portions of \( x \)-circles lying inside the \( |\Gamma| = 1 \) boundary are shown in dashed lines in Fig 1. The following is a list of several salient properties of the \( x \)-circles:

1. The centers of all \( x \)-circles lie on the \( \Gamma_r = 1 \) line: those for \( x > 0 \) (inductive reactance) lie above the \( \Gamma_r \)-axis and those for \( x < 0 \) (capacitive reactance) lie below the \( \Gamma_r \)-axis.
2. The \( x = 0 \) circle becomes the \( \Gamma_r \)-axis.
3. The \( x \)-circles become progressively smaller as \( |x| \) increases from 0 toward \( \infty \), ending at the \(( \Gamma_r = 1, \Gamma_i = 0 \) point.
4. All the \( x \)-circles pass through the \(( \Gamma_r = 1, \Gamma_i = 0 \) point.

A Smith chart is a chart of \( r \) and \( x \) circles in the \( \Gamma_r - \Gamma_i \) plane for \( |\Gamma| \leq 1 \). It can be proved that the \( r \)- and \( x \)-circles are everywhere orthogonal to one another. The intersection of an \( r \)- and an \( x \)-circles defines a point that represents a normalized load impedance \( z_L = r + jx \). The actual load impedance is \( Z_L = R_0 \ ( r + jx \) ). Since a Smith chart plots the normalized impedance, it can be used for calculations concerning a lossless transmission line with an arbitrary characteristic impedance.

As an illustration, point \( P \) in Fig. 1. is the intersection of the \( r = 1.7 \) circle and the \( x = 0.6 \) circle. Hence it represents \( z_L = 1.7 + j0.6 \). The point \( P_{sc} \) at \(( \Gamma_r = -1, \Gamma_i = 0 \) corresponds to \( r = 0 \) and \( x = 0 \) and, therefore, represent a short-circuit. The point \( P_{oc} \) at \(( \Gamma_r = 1, \Gamma_i = 0 \) ) corresponds to an infinite impedance and represent an open-circuit.

The Smith chart in Fig. 1 marked with \( \Gamma_r \) and \( \Gamma_i \) rectangular coordinates. The Smith chart can be marked with polar coordinates,
such that every point in the $\Gamma$-plane is specified by a magnitude $|\Gamma|$ and a phase angle $\theta_\Gamma$. This is illustrated in Fig 2, where several $|\Gamma|$-circles are shown in dotted lines and some $\theta_\Gamma$-angles are marked around the $|\Gamma| = 1$ circle. The $|\Gamma|$-circles are normally not shown on commercially available Smith charts: but once the point representing a certain $z_L = r + jx$ is located, it is a simple matter to draw a circle centered at the origin thorough the point. The fractional distance from the center to the point (compared with the unity radius to the edge of the chart) is equal to the magnitude $|\Gamma|$ of the load reflection coefficient; and the line to the point makes with the real axis is $\theta_\Gamma$.

Each $|\Gamma|$-circle intersects the real axis at two points. In Fig. 2 we designate the point on positive-real axis ($OP_{oc}$) as $P_M$ and the point on the negative-real axis ($OP_{sc}$) as $P_m$. Since $x = 0$ along the real axis, $P_M$ and $P_m$ both represent situations with a purely resistive load, $Z_L = R_L$: obviously $R_L > R_0$ at $P_M$, where $r > 1$; and $R_L < R_0$ at $P_m$, where $r < 1$. We found that $S = R_L/R_0 = r$ for $R_L > R_0$. This relation enables us to say immediately that the value of the $r$-circle passing through the point $P_M$ is numerically equal to the standing-wave ratio. Similarly, we conclude from Eq. (142) that the value of the $r$-circle passing through the point $P_m$ on the negative-real axis is numerically equal to $1/S$. So all the real impedances along the transmission line are as follows:

$$R_{Lm} = \frac{R_0}{S}, \quad R_{LM} = R_0S \tag{142}$$

For the $z_L = 1.7 + j0.6$ point, marked $P$ in Fig. 2, we find $|\Gamma| = 1/3$ and $\theta_\Gamma = 28^\circ$. at $P_M$, $r = S = 2.0$ these results can be verified analytically.
In summary, we note the following features:

1. All $|\Gamma|$-circles are centered at the origin, and their radii vary uniformly from 0 to 1.
2. The angle, measured from the positive real axis, of the line drawn from the origin through the representing $z_L$ equals $\theta_\Gamma$.
3. The value of the $r$-circle passing through the intersection of the $|\Gamma|$-circle and the positive-real axis equals the standing-wave ratio $S$.

So far we have based the construction of the Smith chart on the definition of the voltage reflection coefficient of the load impedance. The input impedance looking toward the load at a distance $z'$ from the load is the ratio of $V(z')$ and $I(z')$. We have, by writing $j\beta$ for $\gamma$ for a lossless line.
The normalized input impedance is

\[
Z_i(z') = \frac{V(z')}{I(z')} = Z_0 \left[ \frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}} \right]
\]  

(143)

The normalized input impedance is

\[
z_i = \frac{Z_i}{Z_0} = \frac{1 + \Gamma e^{-j2\beta z'}}{1 - \Gamma e^{-j2\beta z'}}
\]

(144)

\[
= \frac{1 + |\Gamma| e^{j\phi}}{1 - |\Gamma| e^{j\phi}}
\]

(145)

where

\[
\phi = \theta_{\Gamma} - 2\beta z'
\]

(146)

We note that Eq.(144) relating \(z_i\) and \(\Gamma e^{-j2\beta z'} = |\Gamma| e^{j\phi}\) is of exactly the same form relating \(z_L\) and \(\Gamma = |\Gamma| e^{j\theta_{\Gamma}}\). In fact, the latter is a special case of the former for \(z' = 0\) (\(\phi = \theta_{\Gamma}\)). The magnitude, \(|\Gamma|\), of the reflection coefficient and, therefore, the standing-wave ratio \(S\), are not changed by the additional line length \(z'\). Thus just as we can use the Smith chart to find \(|\Gamma|\) and \(\theta_{\Gamma}\) for a given \(z_L\) at the load, we can keep \(|\Gamma|\) constant and subtract (rotate in the clockwise direction) from \(\theta_{\Gamma}\) an angle equal to \(2\beta z' = 4\pi z' / \lambda\).

This will locate the point \(|\Gamma| e^{j\phi}\), which determines \(z_i\). Two additional scales in \(\Delta z'/\lambda\) are usually provided along the perimeter of the \(|\Gamma| = 1\) circle for easy reading of the phase change \(2\beta(\Delta z')\) due to a change in line length \(\Delta z'\): the outer scale is marked “wavelength towards generator” in the clockwise direction (increasing \(z'\)); and the inner scale is marked “wavelength towards load” in the counterclockwise direction (decreasing \(z'\)). Figure 1.03 is a typical
Smith chart, which is commercially available. It has a complicated appearance, but actually it consists merely of constant-\( r \) and constant-\( x \) circles. We note that a change of half-a-wavelength in line length (\( \Delta z' = \lambda/2 \)) corresponds to a \( 2\beta(\Delta z') = 2\pi \) change in \( \phi \). A complete revolution around a \( |\Gamma| \)-circle returns to the same point and results in no change in impedance.

In the following we shall illustrate the use of the Smith chart for solving some typical transmission-line problems by several examples.

SMITH CHART       APPLICATION

\[
\begin{array}{c}
\text{0.06}\lambda \\
\text{0.12}\lambda \\
\end{array}
\]

\( z\frac{\lambda}{50} \Omega \)

\( z\frac{\lambda}{50} \Omega \)

\( \alpha=0 \)

\( \alpha=0 \)

\( \text{C} \)

\( \text{B} \)

\( \text{A} \)

\( \text{L} \)

a) \( z_x, z_y, z_z, z_m = ? \)

b) Find VSWR, maximum and minimum voltage positions for both lines.

SOLUTION :
Solution is obtained by the Analytical and Graphical methods.

1) Analytical method

\[
Z_L \to \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{z_L - 1}{z_L + 1} \quad (z_L = \frac{Z_L}{Z_0})
\]

\[
z_L = 2 + 1j1.5 \Rightarrow \Gamma_L = \frac{2 + j1.5 - 1}{2 + j1.5 + 1} = \frac{1 + j1.5}{3 + j1.5} = 0.46 + j0.26
\]
\[ Z_L = 23 + j13\Omega \]

\[ Z_A = \frac{V_A}{I_A} = Z_0 \frac{1 + \Gamma_A}{1 - \Gamma_A} = \frac{1 + \frac{\Lambda e^{-j\beta}}{1 - \frac{\Lambda e^{-j\beta}}{1 - \Gamma_A}}}{1 - \Gamma_A} = Z_0 \frac{1 + \Gamma_L e^{-j\frac{4\pi}{0.12\lambda}}}{1 - \Gamma_L e^{-j\frac{4\pi}{0.48\pi}}} \]

\[ Z_B = Z_A + j30 \quad Y_B = \frac{1}{Z_B} \quad Y_C = Y_B + \frac{1}{-j200} \quad Z_C = \frac{1}{Y_C} \]

\[ \Gamma_C = \frac{Z_C - 1}{Z_C + 1} \Rightarrow Z_{in} = Z_0 \frac{1 + \frac{\Lambda e^{-j\frac{4\pi}{0.06\lambda}}}{1 - \frac{\Lambda e^{-j\frac{4\pi}{0.24\lambda}}}{1 - \frac{\Lambda e^{-j\frac{4\pi}{0.24\lambda}}}}}} \]

2) Graphical Method

\[ P_A : Z_A = 1 - j1.3 \]

\[ Z_B = 1 - j1.3 + j\frac{30}{50} = 1 - j0.7 \]
\( P_B : Z_B = 1 - j0.7 \rightarrow \Gamma = 1 \), \( j=0.7 \)

\[ Y_B \rightarrow \text{After taking symmetry of } P_B \text{ according to the origin } Y_B \text{ could be found on the graph} \]

\( P_B^\prime : Y_B = 0.67 + j0.47 \)

\[ Y_C = Y_B + \frac{j}{200} \cdot 50 = 0.67 + j0.72 \]

\( P_C \rightarrow \text{After taking symmetry of } Y_C \text{ according to the origin, } P_C \text{ is found which corresponds to the impedance:} \)

\[ P_C : Z_C = 0.7 - j0.74 \]

\[ Z_{in} = z_0 \cdot z_{in} = 50(0.45 - j0.38)\Omega \]

\( Z_{in} = 22.5-j19\Omega \rightarrow \text{real input impedance} \)

**Example 2**

\[
\begin{align*}
\left(\frac{1}{\lambda}\right)_1 &= \left(\frac{5\text{ cm}}{20}\right) = 0.25\lambda \quad \text{1.Line} \\
\left(\frac{12.8\text{ cm}}{\lambda}\right)_2 &= \left(\frac{12.8\text{ cm}}{20}\right) = 0.64\lambda \quad \text{2.Line}
\end{align*}
\]
Using Smith Chart on Lossy Lines
\[ Z_{in} = \frac{Z_{in}}{Z_0} = \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} = \frac{1 + |\Gamma_L| e^{j\theta_L} e^{-2\alpha d} e^{-j2\beta d}}{1 - |\Gamma_L| e^{j\theta_L} e^{-2\alpha d} e^{-j2\beta d}} \]

Position of the \( \Gamma_L \) on the lossy lines (Lowering of the modules because of lossness of the line)

\[ Z_{in} = 1 + \frac{|\Gamma_L| e^{-2\alpha d} e^{j(\theta_L - 2\beta d)}}{1 - |\Gamma_L| e^{-2\alpha d} e^{j(\theta_L - 2\beta d)}} \]

Formula of the normalized input impedance on lossy line

\[ Z_{line} = \frac{V_{line}}{I_{line}} \]

**Numerical Application**

\[ Z_{L1} = 0, \quad l = 2m, \quad Z_0 = 75\Omega, \quad Z_{in} = 45 + j225\Omega \]

a) \( \alpha, \beta = ? \)

b) \( Z_{L2} = 67.5 - j45\Omega \quad \rightarrow \quad Z_{in} = ? \)
\[
\Gamma_{L1} = \frac{Z_{L1} - Z_0}{Z_{L1} - Z_0} = -1 \implies \Gamma_L = 1e^{j\pi} \\
\Rightarrow \theta_L = \pi \text{ rad}
\]
\[
z_{in} = \frac{Z_{in}}{Z_0} = \frac{45 + j225}{75} = 0,6 + j3
\]
\[
z_{in} = \frac{1+\Gamma_{in}}{1-\Gamma_{in}} = \frac{1+\Gamma_{L}e^{-\gamma(1-z)}e^{-2\alpha d}e^{-j2\beta d}}{1-\Gamma_{L}e^{-\gamma(1-z)}e^{-2\alpha d}e^{-j2\beta d}} \\
d = 1-z, for z = 0
\]
\[
z_{in} = \frac{1+e^{j\pi}e^{-2\alpha d}e^{-j2\beta d}}{1-e^{j\pi}e^{-2\alpha d}e^{-j2\beta d}} = \frac{1+e^{-2\alpha e^{j(\pi-2\beta d)}}}{1+e^{-2\alpha e^{j(\pi-2\beta d)}}} = 0,6 + j3
\]

If the line is lossless input impedance is purely reactive. Also it could be inductive or capacitive. This condition changes by the length of line.
Graphical Solution

$\alpha$ and $\beta$ are found by using the formulas below

$$\left| \frac{OP_{in}}{OP_{in}'} \right| = e^{-2\alpha d}$$

$$\theta = 2\beta l$$

$$\left| \frac{OP_{in}}{OP_{in}'} \right| = e^{-2\alpha d} = 0.89 \Rightarrow \frac{\ln 0.89}{-2l} = 0.028 \frac{Np}{m} = \alpha = 0.25 \text{ dB}$$

(1Np=8.69 dB),

$$Z_{i_1} = 45 + j225\Omega$$

$\alpha = 0.029$, 
$\beta = 0.2\pi$, 
$Z_o = 75\Omega$
b)

\[ Z_{\text{in}} \]

\[ Z_L = \frac{Z_L}{Z_0} = 0.9 - j0.6 \]

0.2\(\lambda\) towards generator

\[ 0.365 + 0.2 = 0.565\lambda \equiv 0.065\lambda \]
Impedance matching is one of the most important subjects of transmission lines. If the characteristic impedance $Z_0$ of the line is equal to the load impedance $Z_L$, the reflection coefficient $\Gamma_L=0$, and the standing wave ratio is unity. When this situation exists, the characteristic impedance of the line and the load impedance are said to be matched, that is, they are equal. In most transmission line applications, it is desirable to match the load impedance to the characteristic impedance of the line in order to reduce reflections standing waves that jeopardize the power-handling capabilities of the line and also distort the information transmitted. Impedance matching is also desirable in order to drive a given load most efficiently (i.e. to deliver maximum load), although maximum efficiency also requires matching the generator to the line at the source end. In the presence of sensitive components (low-noise amplifiers), impedance matching improves the signal-to-noise ratio of the system in other cases generally reduces amplitude and phase errors.

The equivalent circuit is shown below:
P(z) = P^+ \left( 1 - |\Gamma(z)|^2 \right) \Rightarrow \text{the power formula}

1) \Gamma_g=0 \quad \rightarrow \quad Z_g=Z_0 \quad P+=P_{\text{max}} \quad \text{(maximum power transfer)}
2) \Gamma_L=0 \quad \rightarrow \quad Z_L=Z_0 \quad P_L=P+

There are different methods of achieving impedance matching:

1-Matching using series or parallel lumped reactive elements
2-Single stub matching (series or shunt)
3-Double stub matching
4-Triple stub matching

SINGLE STUB SERIES IMPEDANCE MATCHING:

At microwave frequencies, it is often impractical or inconvenient to use lumped elements for impedance matching. Instead, we use a common matching technique that uses single open or short-circuited stubs connected either in series or in parallel. In practice, the short-circuited stub is more commonly used for coaxial and wave-guide applications because a short-circuited line is less-sensitive to external influences (such as capacitive coupling and pick-up) and radiates less than an open-circuited line segment. However, for microstrips and striplines, open-circuited stubs are more common in practice because they are easier to fabricate.

The principle of matching with stubs is similar to matching using lumped reactive elements. The only difference is that the matching impedance (Zs) is introduced by using open or short-circuited line segments at appropriate length (\ell).
In the figure above we can see a short-circuited single stub series impedance matching circuit. Here, we will find out appropriate \( \ell \) and \( d \) lengths that the input impedance of the matching circuit becomes \( Z_0 \) (\( Z_{\text{in}}=Z_0 \)).

As we study at normalized dimensions, following equations can be found:

\[
Z_{\text{in}} = \frac{Z_{\text{in}}}{Z_0} \quad \text{and} \quad Z_L = \frac{Z_L}{Z_0}
\]

\[
z_{\text{in}}' = \frac{(Z_L + j \tan \beta d)}{(1 + j Z_L \tan \beta d)} = 1 + j X_{\text{in}}'
\]

This is the input impedance that is observed from right side of the stub!

\[\text{Re}\{z_{\text{in}}'\}=1 \quad \text{Im}\{z_{\text{in}}'\}=X_{\text{in}}'
\]

The equivalence of the matching circuit is like this:

\[z_{\text{in}} = z_{\text{in}}' + jx = 1 = 1 + j0\]
\[ z_{in} = 1 + jX_{in}' + jx = 1 + j0 \]

\[ X_{in}' + X = 0 \]

\[ X = -X_{in}' \]

So, chosen \( \ell \) and \( d \) lengths must supply these equations.

- Let’s think about pure resistive load impedance (\( Z_L = R \), \( z_L = \overline{R} = r \))

**If \( \tan \beta d = t \), then**

\[ z_{in}' = 1 + j \overline{X_{in}'} = (r + j.t) / (1 + j.r.t) \]

\[ (1 + j \overline{X_{in}'}) \cdot (1 + j.r.t) = (r + j.t) \]

Imaginary and real parts of both sides will be equal:

\[ 1 - \overline{X_{in}'} . r . t = r \]

\[ j(\overline{X_{in}'} + r.t) = j . t \quad \Rightarrow \quad \overline{X_{in}'} = (1 - r).t \]

\[ t = (1 - r) / (1 - r).r.t \quad \Rightarrow \quad t^2 = \tan^2 \beta d = 1/r \]

\[ \tan^2 \beta d = (1 - \cos^2 \beta d) / \cos^2 \beta d = 1/r \]

By this equation, \( d \) can be found like this:

\[ d = (\lambda/4\pi) \cdot \arccos [(r - 1) / (r + 1)] \]

And \( \ell \) can be found as below:

\[ -j \overline{X_{in}'} = -j \cot \beta \ell \quad \Rightarrow \quad \ell = (\lambda/2\pi) \cdot \arctan (\sqrt{r} / 1 - r) \]

- If the load impedance is not pure resistive (\( z_L = rL + xL \)):
Then we look at the max. points of the wave:

\[ r_{\text{max}} = \text{VSWR} = \frac{(1 - |\Gamma_L|)}{(1 + |\Gamma_L|)} = \bar{R} \]

So, new formulas of \( d \) and \( \ell \) are:

\[ d' = \left(\frac{\lambda}{4\pi}\right) \cdot \arccos \left[\frac{(\text{VSWR} - 1)}{(\text{VSWR} + 1)}\right] \quad d = d' + d_{\text{max}} \]

\[ \ell = \left(\frac{\lambda}{2\pi}\right) \cdot \arctan \left(\sqrt{\frac{\text{VSWR}}{1 - \text{VSWR}}}\right) \]

**GRAPHICAL SOLUTIONS:**

Impedance matching problems can be solved easily using the Smith Chart. Let’s look at an antenna matching example:

- To consider stub matching it helps to have a practical example. Here, we study a load formed by an antenna which is being used away from its design frequency. The method is not restricted to antenna loads.

For a 1 metre long dipole antenna at 120 MHz, the load impedance is 44.8 ohms - j 107 ohms. The normalised impedance is 0.597 - j 1.43 with respect to the 75 ohm coaxial line. We shall determine the position and length of a series stub which will match this antenna to the transmission line.

If we look at the **SMITH Chart** we find a circle of constant real normalised impedance \( r=1 \) which goes through the open circuit point and the centre of the chart. In our example in the next picture, this circle is drawn in red. If you plot any arbitrary normalised impedance on the SMITH chart, and follow round clockwise at constant radius, from the centre of the SMITH chart, towards the generator (along the green line in the example), you **must** cross the \( r=1 \) circle somewhere. This transformation at constant radius represents motion along the
transmission line towards the generator. (One complete circuit of the SMITH chart represents a travel of one half wavelength towards the generator.) At this intersection point the generalised arbitrary load impedance \( r + jx \) has transformed to \((1+jx')\), so, at least the real part of the impedance equals the characteristic impedance of the line. Matching has not yet been achieved because of the residual reactance \( x' \) which must be tuned out with the stub. Note that \( x' \) is different from \( x \) in general. For each transformation around the SMITH chart, representing travel one half wavelength towards the transmitter, there are two intersections with the \( r=1 \) circle. Stubs may be placed at either of these points.

At the transformed (see figure –1 ) intersection point (red and green circles) the line is cut and a pure reactance \(-jx'\) is added. This is done by creating this reactance \(-jx'\) using a series-connected lossless stub. Now, the total impedance looking into the sum of the line impedance (which is \(1+jx')\) and \(-jx'\) is therefore \((1+jx') -jx' = 1\) and the line is matched.

Again, one looks at the SMITH chart and finds the outer circle where the modulus of the reflection coefficient is unity. On this circle are the SHORT and OPEN points, and all values of positive (top half of the SMITH chart) and negative (bottom half of the SMITH chart) reactance. The resistance is zero everywhere. It has to be zero, as a lossless transmission line with load infinity ohms (open) or zero ohms (short) has no mechanism for absorbing power. To generate a specified reactance, start at a short circuit (or maybe an open circuit) and follow the rim of the SMITH chart clockwise around towards the generator until the desired reactance is obtained. Cut the stub this number of wavelengths long.
In our example, the SMITH chart construction to find the stub length is shown in the next picture.

From the blue arc in the previous picture we see that the reactance at the r=1 intersection point is +j1.86, so to cancel this out we must add a series stub having reactance -j1.86. In the next figure we plot the blue arc -j1.86 and, starting from the short circuit (r = x = 0) we follow the green line around a distance of 0.328 wavelengths clockwise towards the generator, to generate this value of reactance. If we had started from an open circuit we would only travel a distance (0.328 - 0.250) = 0.078 wavelengths to generate this reactance. This open circuit stub is represented by the red arc.
The practical details of the series stub match are shown in third figure, where we display the physical lengths in centimetres, assuming a wave velocity on the coax (which we need to know to do this calculation) of $2 \times 10^8$ metres per second. This data is supplied by the cable manufacturer. The wave velocity and the frequency (120 MHz) allows us to calculate the wavelength in metres, and thus we can translate the "electrical lengths" from the SMITH chart into physical lengths of line.

\[
\ell = 0.174\lambda \quad \text{and the stub position from load will be } d = 0.47\lambda.
\]
THE ANALYSIS OF THE GENERAL CYLINDRICAL TRANSMISSION LINES

We consider a cylindrical waveguide of arbitrary cross-sectional shape. The longitudinal axis of the waveguide is along the z-direction. The walls of the waveguide are perfect conductors, and the material within the waveguide is characterized by $\varepsilon, \mu$. 

\[
\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} \quad \text{(Faraday)}
\]

\[
\nabla \times \mathbf{H} = (\sigma + j\omega \varepsilon)\mathbf{E} + \mathbf{J}_u \quad \text{(Ampere)}
\]

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon} \quad \text{(Gauss)}
\]

\[
\nabla \cdot \mathbf{H} = 0 \quad \text{(Gauss)}
\]

\[
\mathbf{E}(x, y, z) \\
\mathbf{H}(x, y, z)
\]

\[
\{\nabla^2 + k^2\} \quad \mathbf{D}(x, y, z) = 0 \quad \text{Helmholtz Equation}
\]

\[
\mathbf{B}(x, y, z)
\]

\[
\{J_u=0, \rho=0\}
\]

The Helmholtz equation is a separable linear differential equation. So;

\[
\nabla \mathbf{E}_x(x, y, z) \\
\n\{\nabla^2 + k^2\} \quad \mathbf{E}_y(x, y, z) = 0 \\
\n\mathbf{E}_z(x, y, z)
\]
General Cylindrical Transmission System:

The equation of a wave propagating along the $z$-axis:

$$
\vec{E}(x, y, z) = \vec{E}_r(x, y, z) + \vec{E}_z(x, y, z) = e(x, y)e^{\pm jB_z} + e_z(x, y)e^{\pm jB_z}
$$

$$
\vec{H}(x, y, z) = \vec{H}_r(x, y, z) + \vec{H}_z(x, y, z) = h(x, y)e^{\pm jB_z} + h_z(x, y)e^{\pm jB_z}
$$

As all the EM wave components have to prove the Maxwell Equations, we can analyse these equations for the general cylindrical transmission lines.

Defining the transverse gradient $\nabla_{\tau}$,

$$
\nabla_{\tau} = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y
$$

We have:

$$
\nabla \times \vec{E} = (\nabla_{\tau} + \frac{\partial}{\partial z} a_z) \times \vec{E} = (\nabla_{\tau} - j\beta a_z) \times (e + e_z)e^{-j\beta z}
$$

$$
= -j\omega \mu_0 (h + h_z)e^{-j\beta z}
$$

$$
\nabla_{\tau} \times e + j\beta a_z \times e + \nabla_{\tau} \times e_z - j\beta a_z \times e_z = -j\omega \mu_0 (h + h_z)e^{j\beta z}
$$

Faraday's Law:

$$
\nabla_{\tau} \times e = -j\omega \mu_0 \vec{h}_z
$$

$$
- \vec{a}_z \times \nabla_{\tau} e_z - j\beta \vec{a}_z \times e = -j\omega \mu_0 \vec{h}
$$
Ampere’s Law:
\[ \nabla_t \times \vec{h} = -j \omega \varepsilon e_z \]
\[ \vec{a}_z \times \nabla_t \vec{h}_z + j \beta \vec{a}_z \times \vec{h} = -j \omega \varepsilon e \]

To analyze the general cylindrical transmission lines, first we have to obtain \( \vec{e} \) and \( \vec{h} \) as the parameter of \( e_z \) and \( h_z \).

\[ \vec{e} = g(e_z, h_z) \]
\[ \vec{h} = f(e_z, h_z) \]

Second we have to solve the Helmholtz equation in V domain to obtain \( e_z(x,y) \) and \( h_z(x,y) \) and finally assign all the EM components in V domain.

If we multiply Eq-2 by \(-j \beta \vec{a}_z\) vectorally, we obtain;

\[ -j \beta [\vec{a}_z \times (-\vec{a}_z \times \nabla_t \vec{e}_z) - j \beta \vec{a}_z \times (\vec{a}_z \times \vec{e})] = -j \beta (-j \omega \mu) \vec{a}_z x \vec{h} \]

\[ -j \beta (\vec{a}_z \nabla_t \vec{e}_z)(-\vec{a}_z) + j \beta (\vec{a}_z . (-\vec{a}_z)) \nabla_t \vec{e}_z \]

\[ (k^2 - \beta^2) \vec{e} = j \omega \mu_0 \vec{a}_z \times \nabla_t \vec{h}_z - j \beta (\nabla_t \vec{e}_z) \]

\[ k = \omega \sqrt{\mu_0 \varepsilon} \]

\[ (k^2 - \beta^2) \vec{h} = -j \omega \varepsilon \vec{a}_z \times \nabla_t \vec{e}_z - j \beta (\nabla_t \vec{h}_z) \]
According to these equations, we can separate EM waves propagating along z-direction in cylindrical transmission lines into four groups:

1) **TE (transverse electric) Waves** : $E_z = 0, H_z \neq 0$

2) **TM (transverse magnetic) Waves** : $H_z = 0, E_z \neq 0$

3) **TEM (transverse electromagnetic) Waves** : $H_z = 0, E_z = 0 \Rightarrow \beta = \pm k$

4) **Hybrid** $\Rightarrow E_z \neq 0, H_z \neq 0$

\[
\overline{B}_t = \frac{j \omega \mu \varepsilon \overline{E}_z \times \overline{\nabla}_z E_z + j k \overline{\nabla}_z B_z}{k_0^2 - k^2} \tag{3}
\]

\[
\left\{ \nabla_z^2 + (\mu \varepsilon \omega^2 - k^2) \right\} \frac{\overline{E}_z}{\overline{B}_z} = 0 \tag{4}
\]

TE waves are sometimes called H-waves and TM waves are sometimes called E-waves, where the E-wave and H-wave notation refers to the field that has a z-component. It is important to realize that TE and TM modes are independent solutions, i.e., they independently satisfy Eq (4) and the boundary conditions at the walls. (A solution where both $E_z \neq 0$ and $B_z \neq 0$ would not be an additional independent solution, but rather, if it existed it could be constructed from a superposition of degenerate TE and TM modes. However as we shall now see, the fields for TE and TM modes satisfy different boundary conditions. Consequently, they will not be degenerate.)

For TM waves, the boundary condition that the tangential component of E vanishes at the walls means that $E_z$ vanishes at the walls. This single BC uniquely determines the solution of Eq (4) for TM waves. Therefore, it is unnecessary in the case of TM waves to
impose the other boundary condition at the walls, namely, that the normal component of the magnetic field \((\hat{e}_n \cdot \vec{B})\) in this case) vanishes there. The latter condition must be automatically contained in Eq (3) for \(\vec{B}\) when it is applied to TM waves by setting \(B_z = 0\) in the RHS of that equation. To see this, note that the only component of \(\nabla_t E_t\) that is relevant for finding the normal component of \(B_t\) from Eq (3) is the gradient of \(E_z\) with respect to the coordinate along the boundary, and this vanishes since \(E_z\) is constant there (actually, \(E_z = 0\) at the walls).

For TE waves there is no \(E_z\), thus to solve Eq (4) we use the BC that the normal component of \(\vec{B}\) vanishes at the walls. The latter BC turns out to be equivalent to the condition that the normal derivative of \(B_z\) vanishes at the walls. To see this, calculate \(\hat{e}_n \cdot \vec{B}\) using Eq (3) for \(B_t\). Noting that \(E_z = 0\) for TE waves, we find that \(\hat{e}_n \cdot \vec{B}\) is proportional to \(n \cdot \nabla_t B_z\), which is identical to the normal derivative \(\partial B_z / \partial n\). Thus \(\partial B_z / \partial n\) vanishes at the walls for a TE wave. No other boundary condition is needed to obtain a unique solution of Eq (4) for TE waves. Therefore, the other boundary condition, namely that the tangential component of \(\vec{E}_t\) vanishes at the walls, must be automatically satisfied by this solution for TE waves. (This is easily shown by an argument analogous to that given in the previous paragraph for the case of TM waves.)

**Maxwell Equations in Divergence Form**

\[
\nabla \vec{B} = \nabla \mu \vec{H} = 0 \quad \nabla \vec{H} = 0
\]

\[
(\nabla_t - j \beta a_z)(\vec{h} + \vec{h}_z)e^{-j\beta z} = 0
\]

\[
\nabla_t \vec{h} - j \beta \vec{h}_z = 0 \quad \nabla_t \vec{h} = j \beta \vec{h}_z
\]

\[
\nabla \vec{D} = 0 \quad \nabla_t \vec{e} = -j \beta \vec{e}_z
\]
Obtaining $E_z(x,y)$ ve $H_z(x,y)$:

$$\{\nabla^2 + k^2\} \frac{\overline{E}}{B} = 0 \quad \Rightarrow \quad \{\nabla^2 + k^2\} \frac{\overline{E}_z}{H_z} = 0 \quad (5)$$

$$\nabla^2 = \nabla \cdot \nabla = (\nabla_t - j \beta \overline{a_z})(\nabla_t - j \beta \overline{a_z})$$

$$\nabla^2 = \nabla_t^2 - \beta^2 \quad (6)$$

$$\nabla_t = \frac{\partial}{\partial x} \overline{a_z} + \frac{\partial}{\partial y} \overline{a_y} \quad \nabla_t^2 = \nabla_t \cdot \nabla_t = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Rightarrow (5) \text{ we obtain; }$$

$$\{\nabla^2 + k^2\} \frac{\overline{E}_z(x,y,z)}{H_z(x,y,z)} = 0 \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2} e_z + \frac{\partial^2}{\partial y^2} e_z + \frac{(k^2 - \beta^2)e_z}{h^2 - \lambda^2 - \beta^2} = 0 ;$$

The solution of this differential equation at $E_z=0$, gives $E_z(x,y)$.

$$h^2 = k^2 - \beta^2 \quad \Rightarrow \text{ Characteristic value}$$

$$k^2 = \left( \frac{\omega}{U_\epsilon} \right) , \quad U_\epsilon = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{C}{\varepsilon_r}$$

“$h$” is the function of the problem’s geometry and takes discrete values. We can obtain “$h$” from the solution of the Helmholtz equation for geometry of the problem. So we obtain;

$$\beta = \mp \sqrt{k^2 - h^2} \quad \{\beta \in \mathbb{R} \text{ and } k^2 > h^2 \}$$
For propagation of EM waves $\beta$ must be a member $R$. Using $\beta = \pm \sqrt{k^2 - h^2}$ equation we can analyze the propagation for different conditions of $\beta$.

1) For $\beta = 0$ $k = k_{cutoff} = h$ , $k_c = \frac{\omega_c}{U_\varepsilon} = h$

$$\omega_c = h U_\varepsilon \quad \Rightarrow \quad f_c = \frac{h}{2\pi \sqrt{\mu_0 \varepsilon}} = \frac{h.c}{2\pi \sqrt{\varepsilon_r}}$$

If the EM wave frequency is equal to cutoff frequency, then $\beta = 0$. So no propagation is available.

2) For $k > h$ $\iff$ $k > k_c$ $\iff$ $f < f_c$

$$k^2 - h^2 > 0 \quad \Rightarrow \quad \beta = k \sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

$$\Rightarrow \quad \frac{h}{k} = \frac{k_c}{k} = \frac{\omega_c}{U_\varepsilon} = \frac{U_\varepsilon}{\omega} = \frac{\omega_c}{\omega}$$

3) For $k < h$ $\iff$ $f < f_c$

$$\Rightarrow \quad \beta = \pm \sqrt{k^2 - h^2} = \pm \sqrt{-h^2 \left(1 - \frac{k^2}{h^2}\right)}$$

$\beta$ is imaginary, and causes attenuation.

**Summary**: General cylindrical wave guides have cut off characteristic.
If \( f = f_c \) cut off

If \( f > f_c \) propagation

If \( f < f_c \) attenuation

For \( \omega > \omega_c \):

\[
\left( \frac{\omega}{U_\varepsilon} \right)^2 - \beta^2 = \left( \frac{\omega_c}{U_\varepsilon} \right)^2
\]

For \( \omega < \omega_c \):

\[
\left( \frac{\omega}{U_\varepsilon} \right)^2 + \alpha^2 = \left( \frac{\omega_c}{U_\varepsilon} \right)^2
\]

**RECTANGULAR WAVEGUIDES**

The solution of the EM waves propagating in the ±z direction in the section \( \sigma \) in the systems with only one conductor, the TEM mode cannot exist.
First we must find \( e_z \) and \( h_z \)

**TE WAVES** \( \Leftrightarrow \varepsilon_z = 0 \) \( h_z \neq 0 \)

\[
\{ \nabla t^2 + RC^2 \} h_z = 0 \quad \text{kc}^2 = k^2 - \beta^2 = h^2
\]

\[
\frac{\partial^2}{\partial x^2} h_z + \frac{\partial^2}{\partial y^2} h_z + kc^2 h_z = 0
\]

(1)

\( h_z(x,y) = f(x) \cdot g(y) \)  

(2)

If we put (2) into (1) and divide with \( f \cdot g \)

\[
\frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} + kc^2 = 0
\]

(3)

Only the function of \( x \)  
only the function of \( y \)

\[-kx^2 - ky^2 + kc^2 = 0\]

(4.1)

\[
\frac{1}{f} \frac{d^2 f}{dx^2} - kx^2 \Rightarrow \frac{d^2 f}{dx^2} + kx^2 f = 0
\]

(4.2)

\[
\frac{1}{g} \frac{d^2 g}{dy^2} - ky^2 \Rightarrow \frac{d^2 g}{dy^2} + ky^2 g = 0
\]

(4.3)

• From (4.1) \( f(x,y) = A_1 \cos kx.x + A_2 \sin kx.x \)  

(5.1)

• From (4.2) \( g(x,y) = B_1 \cos ky.y + B_2 \sin ky.y \)  

(5.1)
**BOUNDARY CONDITIONS**

\[
\frac{\partial h_z}{\partial n}_{\text{boundary}} = 0
\]

\[
\begin{align*}
\frac{\partial h}{\partial x}_{x=0} &= 0 \\
\frac{\partial h}{\partial x}_{x=a} &= 0 \\
\frac{\partial h}{\partial y}_{y=0} &= 0 \\
\frac{\partial h}{\partial y}_{y=b} &= 0
\end{align*}
\]

(16)

\[
\frac{\partial f}{\partial x} = -KxA_1 \sin kx.x + kxA_2 \cos A_2 \cos kx.x \bigg|_{x=0} = 0
\]

for \(x=0\) \hspace{1cm} A_2=0 \hspace{1cm} (7.1)

for \(x=a\) \hspace{1cm} -kxA_1 \sin kx \hspace{0.5cm} a=0

\[
\begin{align*}
kx.a &= m\pi \hspace{0.5cm} m = 0,1 \ldots \\
kx &= \frac{m\pi}{a} \hspace{0.5cm} m = 0,1 \ldots
\end{align*}
\]

(7.2)
\[ \frac{\partial h_z}{\partial y} \bigg|_{y=0} = 0 \]
\[ \frac{\partial g}{\partial x} = -B_1 k_y \sin k_y y + B_2 k_y \cos k_y y = 0 \]

For \( y=0 \) \quad B_2 = 0

(7.3)

For \( y=b \) \quad - B_1 k_y \sin k_y b = 0

\[ k_y b = n\pi \]

\[ k_y = \frac{n\pi}{b} \quad n = 0, 1, \ldots \]

(7.4)

Thus,

\[ h(x,y) = f(x) \cdot g(y) \]

\[ h_z(x,y) = H_{mn} \cdot \cos \frac{m\pi x}{a} \cdot \cos \frac{n\pi y}{b} \]

(8.1)

\[ k_c^2 = k_x^2 + k_y^2 = k m_n^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \]

\[ H_{mn} \triangleq A_1 B_1 \]

(8.2)

\[ W_c = w_{mn} = k_c U \epsilon \]

\[ f_c = f_{mn} = \frac{U \epsilon}{2\pi} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right]^{1/2} \]

There is \( \infty \) TE modes and all of them have different cut off frequency.
There is not EM power of the waves propagating in the ±z direction which belong to

\( f_{\infty} \)

\( f = f_{mn} \quad \text{Temn mode status} \quad \beta_{mn} = 0 \)

\( (9.1) \)

\( f > f_{mn} \)

\[
\Gamma_{mn} = j\beta_{mn} = j\left( k^2 - k_{mn}^2 \right)^{1/2} = j\left[ \left( \frac{w}{U\epsilon} \right)^2 - \left( \frac{w_{mn}}{U\epsilon} \right)^2 \right]^{1/2}
\]

\[
\Gamma_{mn} = j\left[ \left( \frac{2\pi}{\lambda} \right)^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2 \right]^{1/2}
\]

\( m = 0, 1..., n = 0, 1... \)

\( f < f_{mn} \Rightarrow \Gamma_{mn} = \alpha_{mn} = \left( k_{mn}^2 - k^2 \right)^{1/2} \)

\( (9.3) \)

For TM\( mn \) h\( z = 0 \), Ez \( \neq 0 \) (\( \nabla t^2 + kc^2 \)) Ez = 0 are propagation parameters

\[
\text{TM}mn = \text{ez} (x, y) = \text{Emn} \sin \frac{m\pi x}{a}.\sin \frac{n\pi y}{b}
\]

\[
\text{TM}mn = \text{hz} (x, y) = \text{Hmn} \cos \frac{m\pi x}{a}.\cos \frac{n\pi y}{b}
\]

\[
f_{mn} = \frac{U\epsilon}{2} \sqrt{\left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2} \quad m = 0, 1...
\]

\( n = 0, 1..., a > b \)
There is $\infty \times \infty$ number of $TE_{mn}$ and $TM_{mn}$ modes

$$TE_{10} = f_{10} = \frac{U \in}{2a} \quad f_{10} < f < f_{20}$$

One mode frequency band
In practice the circular waveguides are mostly used in dominant mode.
In this way one mode propagation is provided.

$T\varepsilon_{01} \rightarrow TM_{11}$

$T\varepsilon_{20} \rightarrow TM_{11}$

The Lowest Cut Off Frequency is $Te_{10}$

$TE_{10} \rightarrow f_{10} = \frac{U\varepsilon}{2a}$ The Lowest Cut Off Frequency

$\frac{U \in}{2a} < f < \frac{U \in}{a}$ Allowable Operating Frequency Range

$TE_{10}$ mode is the dominant mode for rectangular waveguides. ($a>b$)

$f_{10} = \frac{U \in}{2a} \rightarrow TE_{10} \rightarrow$ The lowest cut off frequency ($a>b$)

$f_{20} \rightarrow TE_{20} \rightarrow$ second lowest cut off frequency
*In commercial waveguides (a=2b)
*In TM mode m= 0 n = 0 is not possible

The design of rectangular waveguides for a given frequency.

\[ \lambda \in \frac{2}{U} < a < \lambda \in \frac{U}{f} \]

\[ f = 1.6 \text{ Hz} \Rightarrow \lambda \in \frac{3.10^8}{10^9} = 0.3m = 30 \text{ cm} \]

15 cm < a < 30 cm

**THE WIDTHWISE EM COMPONENTS FOR TE_{mn} AND TM_{mn} MODES**

\[ H_t = \frac{j\beta}{k} \nabla t H_z = \beta / kc \left( \frac{\partial h_z}{\partial x} a_z + \frac{\partial h_z}{\partial y} a_y \right) e^{-j\beta z} \]

\[ U_g = (d\beta / dw)^{-1} \]

\[ E_t = -j\beta / k c^2 \nabla t E_z \]

**PROPAGATION SPECIALITIES**

\[ \beta = \frac{2\pi f}{U} \sqrt{1 - \left( \frac{f_{mn}}{f} \right)^2} \rightarrow \beta = \sqrt{k^2 - kmn^2} = \]

wt-\(\beta z=k\) the speed of constant phase lane
\[ U_p = \frac{w}{\beta} = \frac{U \in}{\sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2}} > U \in \]

For general rectangular waveguide the speed of waves are bigger than the speed in space

\[ F_{mn} \rightarrow \text{the cutoff frequency for } \text{TE}_{mn} \text{ or } \text{TM}_{mn} \]

GROUP SPEED

\[ U_g = \frac{1}{d\beta/dw} = U \varepsilon \sqrt{1 - \left(\frac{f_{mn}}{f}\right)^2} < U \in \]
Ug.Up = Uε^2 \quad Up > Ug

**POWER**

\[
\bar{P}_{ort} = \frac{1}{2} \text{Re}\left\{\bar{E}\bar{H}^*\right\}W/m^2
\]

\[
\bar{P}_{ort} = W_{ort} U \Leftrightarrow U \epsilon = \frac{P_{ort}}{W_{ort}} = Ug
\]

**Guided wave length**

\[
\lambda_g = \frac{2\pi}{\beta} = \frac{\lambda \epsilon}{\sqrt{1 - (f_{mnk} l_f)^2}} > \lambda \epsilon
\]

\[
\lambda \epsilon = \frac{2\pi}{k} = \lambda_g \cdot \beta = 2\pi
\]

The guided waves wave length decreases.
WAVE IMPEDANCES \( Z_{TE}, Z_{TM} \)

\[
Z_{TEMN} = \frac{2}{\sqrt{1 - \left( \frac{f_{mn}}{f} \right)^2}} > \eta \quad \eta = \frac{377}{\sqrt{\varepsilon r}}
\]

\[
Z_{TM} = \eta \sqrt{1 - \left( \frac{f_{mn}}{f} \right)^2} < \eta
\]

\[
Z_{Te}^{TM} = \frac{E_t}{Ht} \quad \eta = \frac{E}{H}
\]

az: The direction of EM power propagation

CALCULATION OF \( P_{mn} \) (For \( TE_{mn} \) and \( TM_{mn} \))

\[
P_{mn} = \int \int \vec{P}_{on} \vec{d}s \rightarrow \text{The net power propagating in the z direction}
\]
\[ P_{mn} = \frac{1}{2} \text{Re} \int_{x=0}^{a} \int_{y=0}^{b} (\bar{E}, x \bar{H}^{\ast}) \bar{a} z \, dx \, dy \]

\[ = \frac{1}{2} \text{Re} \int_{0}^{a} \int_{0}^{b} [E_x H^{\ast} y - E_y H^{\ast} x] \, dx \, dy \]

\[ = \frac{1}{2} \text{Re} \int_{0}^{a} \int_{0}^{b} [H_y H^{\ast} y + H_x H^{\ast} x] \, dx \, dy \]

\[ Z_{mn} = \frac{E_{xmn}}{H_{ymn}} = \frac{E_{ymn}}{H_{xmn}} \]

\[ P_{mn} = \frac{1}{2} \text{Re} \ Z_{mn} \int_{\text{widthwise section}} \int |H_x|^2 + |H_y|^2 \, dx \, dy \]

By using \( H_x \) and \( H_y \)

\[ P_{mn} = \frac{1}{2} \text{Re} \ Z_{mn} \int_{0}^{a} \int_{0}^{b} (\sin^2 \frac{m \pi}{a} x + \cos^2 \frac{n \pi y}{b} \) \, dx \, dy \]

\[
\begin{cases}
ab \frac{4}{n \neq 0} & m \neq 0 \\
ab \frac{2}{2} & n = 0 & m \neq 0
\end{cases}
\]
Z\omega mn |Ht|^2 = Z\omega mn \frac{|Et|^2}{Z\omega n^2} = \frac{|Et|^2}{Z\omega n}

TOTAL EM POWER FOR TE_{mn} or TM_{mn} MODES

\[ P_{mn} = \frac{|Hmn|^2 \ ab}{2. \varepsilon \ om \in \varepsilon \ on} \]

Here \varepsilon \ om and \varepsilon \ on are NEUMAN FACTORS

\[ \varepsilon \ om = \begin{cases} 1 & m = 0 \\ 2 & m > 0 \end{cases} \]

\[ \varepsilon \ on = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases} \]

\[ \beta_{10} = \sqrt{k^2 - k_{10}^2} \quad k_{10} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

FOR T\varepsilon_{10} MODE

\[ P_{10} = \frac{1}{4} w\mu \beta_{10} \left(\frac{a}{\pi}\right)^2 \ ab \ |H_{10}|^2 \]

\[ P_{10} = \frac{1}{4} \frac{1}{ZT \ v_{10}} \ ab \ |E \ max|^2 \]

\[ E_m o x = \frac{w\mu o a H o}{\pi} \]
TE_{10} \rightarrow E_T = E_y \text{ ay} \quad \text{Total field is only in the y direction.}

M=1 \ n=0

(DOMINANT MODE)

Emox < Edielectric distortion

From TE_{10} mode \ Ez=0 \quad Hz\neq0 \quad Ex=0 \text{ can be find}

E \ is \ only \ at \ y \ direction \ and \ at \ x= a/2 \text{ there is maximum Ey}

The electrical fields is maximum at x=a/2. In other regions the change is \sin \pi/.

\[ Ey = -jw \mu_0 \frac{a}{\pi} H_{10} \sin \frac{\pi}{a} x e^{-j\beta} \]
\[ H_x = j\beta \frac{a}{\pi} H_{10} \sin \frac{x}{a} e^{-j\beta z} \]

\[ H_z = H_{10} \cos \frac{\pi x}{a} e^{-j\beta z} \]

\[ P_{10} = \frac{1}{4} \frac{1}{Z T E_{10}} a b E_{\text{max}}^2 \]

Emax < Edielectric distortion happens
Emax ≥ Edielectrik distortion doesnot happens

If system is given then Emax can be find and maximum power occurs.

**THE CONDUCTIVITY LOSSES**
\[ P = \frac{1}{2ZW} \int Et H^* t ds = ZW \int Ht.Ht^* \bar{ds} \quad Ht \frac{1}{Zw} (azxEt) \]

\[ \frac{Et}{Ht} = Zw \]

\[-\frac{\partial p}{\partial z} = PL = 2\alpha P \rho e^{-2\alpha} \]

\[ = 2\alpha P = 2(\alpha c + \alpha d)P \]

canductivity dielectric
lass loss

\[ \alpha c = \frac{PL}{2P} \]

\[ P_L = \frac{Rs}{2} \int Ht.Ht^* dl \]

\[ Rs = \frac{1}{\delta_{gs}} \] because of peffective depth there is a \( R_s \) surface impedance

\[ \alpha c = \frac{Rs \int e Ht.Ht dl}{Zw \int_s Ht.Ht ds} \] (NP/M) conductivity loss constant

\[ \alpha c \rightarrow \text{It is the result of ideal material} \]
DIELECTRIC LOSSES

\[ E_{\text{ef}} = \varepsilon - j \frac{\sqrt{d}}{w} d: \text{dielectric conductivity} \]

REMEMBER

\[ \nabla \times \vec{H} = (\delta d + j \omega \varepsilon) \vec{E} \quad J_{\parallel} = 0 \]

\[ \nabla \times \vec{H} = j \nu (\varepsilon - j \frac{\delta d}{w}) \vec{E} \]

\[ \gamma = \alpha d + j \beta = j \sqrt{k^2 - k_c^2} \]

\[ \gamma = j \sqrt{\varepsilon \mu_0} \varepsilon - k_c^2 \]

\[ \gamma = \alpha d + j \beta \text{ and } \omega \mu_0 \sigma d \ll \omega^2 \mu_0 \varepsilon - k_c^2 / \text{ and also with the use of binomial serials.} \]

\[ \alpha d = \frac{\sigma d}{2 \sqrt{\varepsilon}} \left[ \mu_0 \right] \left\{ 1 - \left( \frac{wc}{w} \right) \right\}^{1/2} \frac{N p}{m} \approx wc \]

The loosing factor in \( e^{-\alpha dz} \), \( \alpha d \) is real and positif.

The relationship bteween \( \alpha B/m \) and \( Np/m \) is
\[ \frac{DB/m}{\alpha} = 10 \log_{10} e^{2 \alpha \Delta z} = 8.686 \alpha \]
\[ = \frac{10}{\Delta z} \left[ \log e^{2 \alpha \Delta z} \right] = 20 x \alpha \times \log_{10} e \]

\[ w >> wc \text{ için } \alpha d = \frac{\sqrt{d}}{2} \sqrt{\mu \omega l} e \]

There are two losses. The \( \alpha c \) is because of material not being ideal. The other loss becomes from cutoff frequency.

**CIRCULAR WAVEGUIDE**

This figure illustrates a cylindrical wave guide with a circular cross section of radius \( r \). In view of the cylindrical geometry involved, cylindrical coordinates are most appropriate for the analysis to be carried out. Since the general properties of the modes that may exist are similar to those for the rectangular guide.

\[ \nabla^2 \psi + k_e \psi = 0 \quad \text{Helmholtz Equation} \]
\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \phi^2} + \left( k_e - \beta^2 \right) \psi = 0
\]

(in the circular cylindrical coordinate)

\[
e^{-j\beta z} \psi(r, \phi, z) = R(r) \Phi(\phi) e^{j\beta z}
\]

\[
\frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + h^2 r^2 = -\frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2}
\]

only function of \( r \) only function of \( \phi \)

The left-hand side is a function of \( r \) only, whereas the right-hand side depend on \( \phi \) only. Therefore this equation can hold for all values of the variables only if both sides are equal to some constant \( k^2 \).

\[
\frac{d^2 \Phi}{d\phi^2} + \kappa^2 \Phi = 0 \quad \Rightarrow \quad \Phi(\phi) = A \cos \kappa \phi + B \sin \kappa \phi \equiv C \cos (\kappa \phi + \varphi)
\]

For given \( r, \phi \) and \( 2n\pi + \phi \) represent same point.

For \( \kappa = n \quad n=0,1,2,\ldots \) And \( \phi = 0 \). \( w \)

\[
\Phi(\phi) = C \cos n\phi
\]
\[ r^2 \frac{d}{dr} \left( r \frac{dR}{dr} \right) + (h^2 r^2 - n^2) R = 0 \]

(The Bessel Differential Equation)

\[
\frac{DJ_n(hr)}{EN_n(hr)} = 0
\]

Bessel Function Neumann Function

In order to the function goes to infinite, it should be \( E = 0 \)

\[
\Psi(r, \phi, z) = DJ_n(hr) \cos n\phi \quad e^{-j\beta z} \quad ; \quad \beta^2 = k_\varepsilon^2 - h^2
\]

\[
\Psi \quad \text{(TE)} \rightarrow H_z
\]

\[
\frac{\partial H_z}{\partial n} \bigg|_{\text{equation of boundary}} \equiv \frac{\partial H_z}{\partial r} \bigg|_{r=b} = 0
\]

\( E_z = 0 \quad ; \quad J_n(hb) = 0 \Rightarrow p_{nm} \Rightarrow q_{nm} \)

\( \text{TE}_{11}, \text{TM}_{01}, \text{TE}_{21}, \text{TE}_{01}/\text{TM}_{11} \)

\( \text{TE}: \)

\[ J_n(hb) = 0 \quad \rightarrow \quad J_n(q_{nm}) = 0 \quad \rightarrow \quad q_{nm} = hb \]
\[
f_{TE_{nm}} = \frac{q_{nm} U_\varepsilon}{2\pi b} \quad \rightarrow \quad h = \frac{q_{nm}}{b} = \frac{\omega_c}{U_\varepsilon}
\]

\[
J_n(hb) = 0 \quad \rightarrow \quad J_n(p_{nm}) = 0 \quad \rightarrow \quad p_{nm} = hb
\]

\[
f_{TM_{nm}} = \frac{p_{nm} U_\varepsilon}{2\pi b}
\]

TE and TM cutoff frequencies are different from each other.

Order of the modes w.r.t the cutoff frequencies (from low to high) \( (\beta_{nm} = 0) \)

\[TE_{11}, TM_{01}, TE_{21}, TE_{01}/TM_{11}, \ldots, TE_{31}\]

EXAMPLE:
(a) \( f = 6 \text{ GHz}, 500 \text{ kW continuous wave power} \ l = 30 \text{ feet}, \) choose a traditional (commercial available) circular wave guide,

(b) Order the lowest five cutoff frequencies,

(c) Find out the operation bandwidth for the TE_{11} mode,

(d) Find out the loss,

(e) Find out the maximum wave for electrical field strength And compare it with break down value for the dry air,

(f) If you insert a Teflon disk in the wave guide, in order to have it as invisible what should its thickness be?

SOLUTION:

(a) \( f = 6 \text{ GHz} \); \( f_{c_{TE_{11}}} = \frac{q_{11} U_\varepsilon}{2\pi b} \)

The operation frequency has to be higher than \( f_{c_{TE_{11}}} \) for the safety margin let us choose.
\[
f = 1.25 \times f_{c_{TE_{11}}} \Rightarrow f_{c_{TE_{11}}} = \frac{f}{1.25}
\]

\[
f_{c_{TE_{11}}} < f < f_{c_{TE_{2}}} \Rightarrow 1.25 \times f_{c_{TE_{11}}} \leq f \leq 0.9 f_{c_{TE_{2}}}
\]

Taking

\[
f_{c_{TE_{11}}} = 5 \text{ GHz} \rightarrow f_{c_{TE_{11}}} = \frac{1.841 \times c}{2\pi b} \Rightarrow 2b = 3.5 \text{ cm} = 1.39'' \Rightarrow \text{WC 150}
\]

We choose standard WC 150 from the table of standard circular wave guides.

\[
\text{WC 150} \Rightarrow 2b = 1.5''
\]

Wave guide \[ \text{Circular} \]

for this value \(2b = 1.5''\) we obtain: \(f_{c_{TE_{11}}} = 4.614 \text{ GHz}\)

<table>
<thead>
<tr>
<th>EAI Designation</th>
<th>Inside Dimensions (Inches)</th>
<th>Recommended Frequency Range TE11 Mode GHz</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Diameter</td>
<td>Tolerance + or -</td>
</tr>
<tr>
<td>WC 992</td>
<td>9.915</td>
<td>0.01</td>
</tr>
<tr>
<td>WC 847</td>
<td>8.47</td>
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<tr>
<td>WC 724</td>
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<td>WC 618</td>
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<tr>
<td>WC 528</td>
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<tr>
<td>WC 451</td>
<td>4.511</td>
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<tr>
<td>WC 385</td>
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<tr>
<td>WC 329</td>
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<tr>
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<td>WC 109</td>
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<td>WC 94</td>
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<tr>
<td>WC</td>
<td>0,797</td>
<td>0,0008</td>
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<td>-----</td>
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<td>WC 11</td>
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</tr>
<tr>
<td>WC 9</td>
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<td>0,00025</td>
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</table>

(b) The lowest five cutoff frequencies the WC 150

<table>
<thead>
<tr>
<th>Mode:</th>
<th>TE$_{11}$</th>
<th>TM$_{01}$</th>
<th>TE$_{21}$</th>
<th>TE$<em>{01}/$TM$</em>{11}$</th>
</tr>
</thead>
</table>

(c) The operation bandwidth,

$$1.15f_{c_{TE_{11}}} \leq f \leq 0.95f_{c_{TE_{21}}}$$

**TM$_{01}$:**

$\rightarrow$ E lines

$\rightarrow$ H lines
TM$_{01}$ is not generally used for the second order mode, since this configuration does occur rarely in practice.

**TE$_{11}$:**

\[ \rightarrow \text{E lines} \]

\[ \begin{array}{c}
\text{TE$_{21}$:} \\
\rightarrow \text{E lines}
\end{array} \]

\[(d) \ \alpha_{c_{TE_{nm}}}\]

n: the order of the Bessel function, m: the order of the zeros
\[
\alpha_{c,TE_{nm}} = \frac{8.686}{\sigma \delta b \zeta \sqrt{1 - \left(\frac{f_c}{f}\right)^2}}
\]

For the 30'' propagation distance of ‘Al’ waveguide the loss power = 0.68 dB

\[
\frac{outputpower}{inputpower} = \%85.4, \quad P_{LOSS} = 72.6 \text{ kW}
\]

If the operation frequency \( f \) increases, the variations \( \alpha_c \) as \( dB/m \) are given below:

For the atmosphere pressure, the circular wave guide with the dry air insulator, the maximum pulsive power can be

\[
P_{\max,TE_{11}} = 2.7(2B)^2 \sqrt{1 - \left(\frac{f_c}{f}\right)^2}
\]

\[
P_{\max,TE_{11}} = 3.88 \text{ MW} ;
\]

\[
E_{\max} = \sqrt{\frac{0.5}{3.88}}x29kV = 10405 \text{ W/cm}
\]