PROBLEMS ON EULER DIFFERENCE

Q.1 The input-output differential equation of the circuit in figure 2 is given as:

\[
\frac{d^2 y_a(t)}{dt^2} + \frac{1}{RC} \frac{dy_a(t)}{dt} + \frac{1}{LC} y_a(t) = \frac{1}{RC} \frac{dx_a(t)}{dt}
\]

The transfer function between \( Y_a(s) \) and \( X_a(s) \) can be obtained from the above equations as:

\[
H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}
\]

Using

\[
\Omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = R \sqrt{\frac{C}{L}}
\]

where \( \Omega_0 \) is the centre frequency and \( Q \) is the quality factor of the band-pass filter, we obtain:

\[
H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{\Omega_0}{s^2 + \frac{\Omega_0}{Q}s + \Omega_0}
\]

Consider the analogue filter given in figure 1 which is composed of an A/D, a digital filter and a D/A, where A/D and D/A are assumed to be perfect converters.

The digital filter in figure 1 is to be designed using the backward Euler-difference such that the overall analogue filter would be a simulation of the band-pass filter given in figure 2.

(a) Obtain the input-output difference equation from the differential equation given above. Sketch a realisation for the digital filter giving the expressions of the multipliers in terms of \( R, L, C \) and the sampling period.

(b) Obtain the transfer function \( H(z) \) of the digital filter from the input-output from the difference equation obtained in part (a). Show that \( H(z) \) can also be obtained from \( H_a(s) \) through:

\[
H(z) = H_a \left( \frac{1 - z^{-1}}{T_s} \right)
\]

(c) Let
 Evaluating the values of $|H(e^{j\omega})|$ at $\omega = 0, \pi$ and centre frequency $\omega_o$ show that for the sampling rate of $F_s = 8$ kHz the analogue filter in figure 4 behaves as a band-pass filter with maximum gain at the fundamental frequency (1st harmonic) of the waveform given in figure 3.

**Q.2** The input-output differential equation of the filter circuit in figure 5 is given as:

\[
\frac{d^2 y_a(t)}{dt^2} + \frac{R}{L} \frac{dy_a(t)}{dt} + \frac{1}{LC} y_a(t) = \frac{1}{LC} x_a(t)
\]

The transfer function between $Y_a(s)$ and $X_a(s)$ can be obtained from the above equations as:

\[
H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{1}{s^2 + \frac{R}{L} s + \frac{1}{LC}}
\]

Using

\[
\Omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = \frac{1}{R \sqrt{C}}
\]

where $\Omega_0$ is the resonance frequency and $Q$ is the quality factor of the filter, we obtain:

\[
H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{\Omega_0^2}{s^2 + \frac{\Omega_0^2}{Q} s + \Omega_0^2}
\]

Consider the analogue filter given in figure 1 which is composed of an A/D, a digital filter and a D/A, where A/D and D/A are assumed to be perfect converters.

The digital filter in figure 1 is to be designed using the backward Euler-difference such that the overall analogue filter would be a simulation of the band-pass filter given in figure 5.

(a) Obtain the input-output difference equation from the differential equation given above. Sketch a realisation for the digital filter giving the expressions of the multipliers in terms of $R, L, C$ and the sampling period.
(b) Obtain the transfer function $H(z)$ of the digital filter from the input-output difference equation obtained in part (a). Show that $H(z)$ can also be obtained from $H_a(s)$ through:

$$H(z) = H_a\left(\frac{1 - z^{-1}}{T_s}\right)$$

Let

$$H(z) = \frac{0.032}{1 - 1.768z^{-1} + 0.8z^{-2}}$$

(i) Compute $|H(e^{j\omega})|$ at $\omega = 0, 1.5$ and $\pi$ and plot $|H(e^{j\omega})|$. 

(ii) Choose the sampling rate as $F_s = 50 \text{ kHz}$ and show that the analogue filter in figure 1 behaves as a low-pass filter passing only the fundamental frequency (1st harmonic) of the waveform given in figure 6. (Hint: The frequency spectrum of the waveform in figure 6 contains odd harmonics only.)

Q.3 The input-output differential equation of the filter circuit in figure 7 is given as:

$$\frac{d^2y_a(t)}{dt^2} + \frac{R}{L} \frac{dy_a(t)}{dt} + \frac{1}{LC} y_a(t) = \frac{d^2x_a(t)}{dt^2}$$

The transfer function between $Y_a(s)$ and $X_a(s)$ can be obtained from the above equations as:

$$H_a(s) = \frac{V_o(s)}{V_i(s)} = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Using

$$\Omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = \frac{1}{R \sqrt{C}}$$

where $\Omega_0$ is the resonance frequency and $Q$ is the quality factor, we obtain:

$$H_a(s) = \frac{Y_a(s)}{X_a(s)} = \frac{s^2}{s^2 + \Omega_0^2 s + Q^2}$$
Consider the analogue filter given in figure 1 which is composed of an A/D, a digital filter and a D/A, where A/D and D/A are assumed to be perfect converters.

The digital filter in figure 1 is to be designed using the backward Euler-difference:

\[
\frac{dy_a(t)}{dt} \bigg|_{t=nT_s} = \frac{y(n)-y(n-1)}{T_s}
\]

such that the overall analogue filter would be a simulation of the high-pass filter given in figure 8.

(a) Obtain the input-output difference equation from the differential equation given above. Sketch a realisation for the digital filter giving the expressions of the multipliers in terms of \( \Omega_0 \), \( Q \) and the sampling frequency.

(b) Obtain the transfer function \( H(z) \) of the digital filter from the input-output difference equation obtained in part (a).

(c) Let

\[
H(z) = \frac{(1-z^{-1})^2}{1-1.768z^{-1}+0.8z^{-2}}
\]

(i) For \( \omega = 0.188 \text{ r/sample} \) \( H(e^{j\omega}) \) is computed as 0.94. Evaluate \( H(e^{j\omega}) \) for \( \omega = 0, \pi/2 \text{ and } \pi \) and roughly plot \( H(e^{j\omega}) \).

(ii) Let \( F_s = 50 \text{ kHz} \). Using the plot obtained in part (i) show that the overall analogue filter in figure 1 behaves as a high-pass filter stopping the 50Hz component and passing the other two components of the periodic waveform given by:

\[
x_a(t) = \cos(2\pi 50t) + \cos(2\pi 1500t) + \cos(2\pi 3000t)
\]