IMAGE PROCESSING

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Image Processing

A Revision of Basic Concepts

An image is mathematically represented by:

\[ I(x, y) \]

where

\( x \) is the vertical spatial distance;
\( y \) is the horizontal spatial distance,

both measured from the top left corner, and

\( I(x, y) \) is the brightness of point \((x, y)\).
Image Processing

A typical black-and-white photograph is composed of shades of gray spanning from black to white, and is known as a continuous tone image.

Digital image processing requires discrete pieces of data on a one-by-one basis.

Therefore the continuous tone image must first be chopped into individual points of information.

This “chopping” is referred to as sampling, because we are taking samples of brightness of the photograph at specific locations within it.
Image Processing

As a result of sampling an image is discretized into a square grid of pixels, each of which is labelled with a pair of coordinates, one defining its line number and the other the column number.

Digitising an image involves yet a further step: Discretization (quantization) of the brightness values in order to have a limited number of brightness levels, e.g., for an 8-bit system $256 = 2^8$. 
The discrete pixel numbering convention
Image Processing

The frequency content of any physical signal depends on the rapidity of its change over a certain observation period.

In DSP this period is referred to as record length.

In an image the frequency components are spatial frequencies which relate to the rapidity of change in gray levels over a certain spatial distance.

Since there are two dimensions there are two spatial frequency components, namely $f_x$ and $f_y$. 
Image Processing

When dealing with the digitisation of an image, there is always the question of how good the representation is when compared with the original.

We define the limitations of the digitisation process with the term resolution.

When we speak of spatial resolution, we are describing how many pixels our digital image is divided into.
An image is said to be composed of many basic frequency components, ranging from low to high.

Rapid brightness transitions give rise to high spatial frequency and slow transitions to low spatial frequency.

Wherever a sharp edge is present—say, a transition from black to white within a one pixel distance—the highest frequencies in the image are found.
Image Filtering

- Making use of this information, we may generate output images showing only, e.g., high-frequency or low frequency components, a class of operation known as spatial filtering.

- Additional spatial filtering operations make it possible to generate images that show only where individual sharp transitions occur. These processes ultimately yield image edge detection and enhancement.
Image Filtering

Linear filtering of an image is accomplished through an operation called **convolution**.

In convolution, the value of an output pixel is computed as a **weighted sum of neighbouring pixels**.

The matrix of weights is called the **convolution kernel**, also known as the **filter**.
Image Filtering

Example:

\[ A = \begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix} \quad h = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \]

Input Image

Convolution Kernel
Image Filtering

Computing the (2,4) output pixel:

1. Rotate the convolution kernel 180 degrees about its centre element.

2. Slide the centre element of the convolution kernel so that it lies on top of (2,4) element of A.

3. Multiply each weight in the rotated convolution kernel by the pixel of A underneath.

4. Sum up the individual products from step 3.
Image Filtering

\[
A = \begin{bmatrix}
17 & 24 & 1 & 8 & 15 \\
23 & 5 & 7 & 14 & 16 \\
4 & 6 & 13 & 20 & 22 \\
10 & 12 & 19 & 21 & 3 \\
11 & 18 & 25 & 2 & 9 \\
\end{bmatrix}
\]

\[
h = \begin{bmatrix}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2 \\
\end{bmatrix}
\]

Values of rotated convolution kernel

Centre of kernel

Input image pixel values

The (2,4) output pixel value is:

\[
13 \cdot 2 + 8 \cdot 9 + 15 \cdot 4 + 7 \cdot 7 + 14 \cdot 5 + 16 \cdot 3 + 13 \cdot 6 + 20 \cdot 1 + 22 \cdot 8 = 575
\]
Image Filtering

This is called either of the following:

• Template
• Image mask
• Convolution kernel
• Impulse response

\[ h = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \]
Image Filtering

Examples of lowpass filter kernels:

\[
\begin{pmatrix}
1/10 & 1/10 & 1/10 \\
1/10 & 2/10 & 1/10 \\
1/10 & 1/10 & 1/10
\end{pmatrix}
\quad \begin{pmatrix}
1/10 & 1/10 & 1/10 \\
1/10 & 2/5 & 1/10 \\
1/10 & 1/10 & 1/10
\end{pmatrix}
\]
Image Filtering

Frequency Response

\[
\begin{bmatrix}
1/10 & 1/10 & 1/10 \\
1/10 & 2/10 & 1/10 \\
1/10 & 1/10 & 1/10
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/10 & 1/10 & 1/10 \\
1/10 & 2/5 & 1/10 \\
1/10 & 1/10 & 1/10
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/10 & 1/10 & 1/10 \\
1/10 & 1/10 & 1/10
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/10 & 1/10 & 1/10
\end{bmatrix}
\]
Image Sampling and the Spatial Frequency Concept

Let us once again repeat what we have already said earlier:

In an image the frequency components are spatial frequencies which relate to the rapidity of change in gray levels over a certain spatial distance.

Since there are two dimensions there are two spatial frequency components, namely \( f_x \) and \( f_y \).

Now consider the image on the next page. The size of this image is 128x128 pixels. Each of the black or white stripes is one pixel wide. The following image is of the same size whose stripes are 4 pixels wide.
Image Sampling and the Spatial Frequency Concept
Image Sampling and the Spatial Frequency Concept
Image Sampling and the Spatial Frequency Concept

Now bearing in mind that a pair of black and white stripes represents one cycle in the horizontal direction, the first figure has 64 cycles and the second contains 16. Therefore the frequency of change in the gray levels, namely, the spatial frequency, in the first one is four times larger than that in the second one.

Now consider the second image. If we draw a straight line in the horizontal direction one can record the change in the gray levels on this line in the form a graph as in the following:
Now let us shift the ordinate axis by a quarter of a period in order to make the phase shift zero.
The Fourier series expansion of $I(x)$ is given as:

$$I(x) = 4\left(\frac{1}{\pi} \cos \Omega_0 x - \frac{1}{3\pi} \cos 3\Omega_0 x + \frac{1}{5\pi} \cos 5\Omega_0 x + \ldots\right)$$
Image Sampling and the Spatial Frequency Concept

The fundamental period of this waveform is given as:

\[ X_0 \text{ units} \] \[ \text{cycle} \]

It is obvious that \( X_0 \) shows a dimension on the image.

The fundamental frequency is obtained as:

\[ F_0 = \frac{1}{X_0 \text{ unit}} \] \[ \Omega_0 = \frac{2\pi \text{ radians / cycle}}{X_0 \text{ unit / cycle}} = \frac{2\pi \text{ radians}}{X_0 \text{ unit}} \]
Assume that the image has the size 0.256mx0.256m. We can write:

\[
X_0 = \frac{0.256}{16} = 0.016 \frac{m}{cycle}
\]

\[
F_0 = \frac{1}{X_0} = 62.5 \frac{cycles}{m}
\]

\[
\Omega_0 = 125\pi \ r / m
\]

This results in the following:

\[
I(x) = 4\left(\frac{1}{\pi} \cos 125\pi x - \frac{1}{3\pi} \cos 375\pi x + \frac{1}{5\pi} \cos 625\pi x + \ldots\right)
\]
Image Sampling and the Spatial Frequency Concept

MATLAB program for plotting $I(x)$ with continuous $x$

```matlab
close all
x=0:1/25000:2/62.5;
I1=(4/pi)*cos((125*pi)*x);I3=(4/(3*pi))*cos((3*125*pi)*x);
I5=(4/(5*pi))*cos((5*125*pi)*x);I7=(4/(7*pi))*cos((7*125*pi)*x);
I9=(4/(9*pi))*cos((9*125*pi)*x);I11=(4/(11*pi))*cos((11*125*pi)*x);
I13=(4/(13*pi))*cos((13*125*pi)*x);I15=(4/(15*pi))*cos((15*125*pi)*x);
I17=(4/(17*pi))*cos((17*125*pi)*x);I19=(4/(19*pi))*cos((19*125*pi)*x);
I21=(4/(21*pi))*cos((21*125*pi)*x);I23=(4/(23*pi))*cos((23*125*pi)*x);
I25=(4/(25*pi))*cos((25*125*pi)*x);I27=(4/(27*pi))*cos((27*125*pi)*x);
I29=(4/(29*pi))*cos((29*125*pi)*x);I31=(4/(31*pi))*cos((31*125*pi)*x);
I=I1-I3+I5-I7+I9-I11+I13-I15+I17-I19+I21-I23+I25-I27+I29-I31;
figure,plot(I);title('Two periods of $I(x)$ drawn using 31 harmonics');
```
Image Sampling and the Spatial Frequency Concept

One period of $f(x)$ drawn using 31 harmonics
Image Sampling and the Spatial Frequency Concept

Sampling of $I(x)$ implies the substitution:

$$x = nX_s$$

where $n$ is an integer and $X_s$ represents sampling interval.

It is obvious that $X_s$ shows a dimension on the image.

Sampling of $I(x)$ implies:

$$I(x)\bigg|_{x=nX_s} = I(nX_s)$$
Image Sampling and the Spatial Frequency Concept

White = +1, Black = -1

128 pixels

White = +1, Black = 0
Image Sampling and the Spatial Frequency Concept

Applying this to the given $I(x)$ we obtain:

$$I(nX_s) = I(n) = 4\left\{\frac{1}{\pi} \cos \Omega_0 X_s n - \frac{1}{3\pi} \cos 3\Omega_0 X_s n + \frac{1}{5\pi} \cos 5\Omega_0 X_s n + \ldots\right\}$$

Now let us have a closer look at $\Omega_0 X_s$. We have already established that:

$$\Omega_0 = \frac{2\pi}{X_0}$$

where $X_0$ is the fundamental period of this waveform.

Now using this in $\Omega_0 X_s$ we can write:

$$\Omega_0 X_s = 2\pi \frac{X_s}{X_0}$$
Image Sampling and the Spatial Frequency Concept

\[
I(n) = 4\left( \frac{1}{\pi} \cos 2\pi \frac{X_s}{X_0} n - \frac{1}{3\pi} \cos 6\pi \frac{X_s}{X_0} n + \frac{1}{5\pi} \cos 10\pi \frac{X_s}{X_0} n + \ldots \right)
\]

Now consider the 128x128 pixel image having 16 periods of the square wave in the horizontal direction. In this case

The sampling interval is given as:

\[
X_s = \frac{0.256}{128} = 0.002 \frac{m}{\text{sample}}
\]

The fundamental period of the waveform is given as:
Image Sampling and the Spatial Frequency Concept

\[ X_0 = \frac{0.256}{16} = 0.016 \frac{m}{cycle} \]

Hence we obtain:

\[ \frac{X_s}{X_0} = \frac{0.002}{0.016} = \frac{1}{8} \]

which yields

\[ I(n) = 4\left(\frac{1}{\pi} \cos \frac{\pi}{4} n - \frac{1}{3\pi} \cos \frac{3\pi}{4} n + \frac{1}{5\pi} \cos \frac{5\pi}{4} n + \ldots\right) \]
Using
\[ F_s = \frac{1}{X_s} \text{ samples} \quad \text{and} \quad F_0 = \frac{1}{X_0} \text{ cycles} \]
we have
\[ I(n) = 4 \left( \frac{1}{\pi} \cos 2\pi \frac{F_0}{F_s} n - \frac{1}{3\pi} \cos 6\pi \frac{F_0}{F_s} n + \frac{1}{5\pi} \cos 10\pi \frac{F_0}{F_s} n + \ldots \right) \]

For the above example we get
\[ F_s = \frac{1}{X_s} = 500 \frac{\text{samples}}{m}, \quad F_0 = \frac{1}{X_0} = 62.5 \frac{\text{cycles}}{m} \]
A much simpler approach to determining dimensions on the image is taking the sampling interval as 1 pixel long. In this case all other dimensions are given by the number of pixels that fall into that dimension. In the case of previous example:

\[ X_s = 1 \text{ pixel}, \quad X_0 = 8 \text{ pixel}, \quad \frac{X_s}{X_0} = \frac{1}{8} \]

It is evident that this ratio decreases as the sampling rate increases, i.e., the number of pixels taken from the image increases. We can make the following MATLAB programs to compute the Fourier series expansion of the horizontal changes in the image for four different values of the sampling rate.
Image Sampling and the Spatial Frequency Concept

MATLAB Program for 128x128 size image, i.e., \( F_s = 500 \text{samples/m} \) or 8 pixels (samples) per period \( X_0 \)

```matlab
close all
n=0:1:7
x1=(4/pi)*cos((pi/4)*n);
x3=(4/(3*pi))*cos((3*pi/4)*n);
x=x1-x3
stem(x1);
figure, stem(x3);
figure, stem(x); title('One period of 128x128 size image I(n) ');
```
Image Sampling and the Spatial Frequency Concept

One period of 128x128 size image \( I(n) \)
Image Sampling and the Spatial Frequency Concept

- close all
- n=0:1:7
- x1=(4/pi)*cos((pi/4)*n);
- x3=(4/(3*pi))*cos((3*pi/4)*n);
- x5=(4/(5*pi))*cos((5*pi/4)*n);
- x7=(4/(7*pi))*cos((7*pi/4)*n);
- x9=(4/(9*pi))*cos((9*pi/4)*n);
- x11=(4/(11*pi))*cos((11*pi/4)*n);
- x13=(4/(13*pi))*cos((13*pi/4)*n);
- x15=(4/(15*pi))*cos((15*pi/4)*n);
- x17=(4/(17*pi))*cos((17*pi/4)*n);
- x19=(4/(19*pi))*cos((19*pi/4)*n);
- x21=(4/(21*pi))*cos((21*pi/4)*n);
- x23=(4/(23*pi))*cos((23*pi/4)*n);
- y=x1-x3
- x=x1-x3+x5-x7+x9-x11+x13-x15+x17-x19+x21-x23
- subplot(2,1,1),stem(y);
- %figure,stem(y);title('One period of 128x128 size image I(n) ');
- subplot(2,1,2),stem(x);
- %figure,stem(x);title('One period of 128x128 size image I(n) ');
Image Sampling and the Spatial Frequency Concept

[Graph showing frequency responses]
Image Sampling and the Spatial Frequency Concept

MATLAB Program for 256x256 size image, i.e., Fs=1000 samples/m or 16 pixels (samples) per period

```matlab
close all
n=0:1:15
x1=(4/pi)*cos((pi/8)*n);
x3=(4/(3*pi))*cos((3*pi/8)*n);
x5=(4/(5*pi))*cos((5*pi/8)*n);
x7=(4/(7*pi))*cos((7*pi/8)*n);
x=x1-x3+x5-x7
stem(x1);
figure,stem(x3);
figure,stem(x5);
figure,stem(x7);
figure,stem(x); ,title('One period of 256x256 size image I(n) ');
```
Image Sampling and the Spatial Frequency Concept
Image Sampling and the Spatial Frequency Concept

MATLAB Program for 512x512 size image, i.e., $F_s=2000$ samples/m or 32 pixels (samples) per period

close all
n=0:1:31
x1=(4/pi)*cos((pi/16)*n);
x3=(4/(3*pi))*cos((3*pi/16)*n);
x5=(4/(5*pi))*cos((5*pi/16)*n);
x7=(4/(7*pi))*cos((7*pi/16)*n);
x9=(4/(9*pi))*cos((9*pi/16)*n);
x11=(4/(11*pi))*cos((11*pi/16)*n);
x13=(4/(13*pi))*cos((13*pi/16)*n);
x15=(4/(15*pi))*cos((15*pi/16)*n);
x=x1-x3+x5-x7+x9-x11+x13-x15
stem(x1);
figure,stem(x3);
figure,stem(x5);
figure,stem(x7);
figure,stem(x9);
figure,stem(x11);
figure,stem(x13);
figure,stem(x15);
figure,stem(x); title('One period of 512x512 size image I(n) ');}
Image Sampling and the Spatial Frequency Concept
Image Sampling and the Spatial Frequency Concept

MATLAB Program for 1024x1024 size image, i.e., $F_s=4000\text{samples/m}$ or 64 pixels (samples) per period

close all

$n=0:1:64$

$x_1=(4/\pi)\cos((\pi/32)*n);$
$x_3=(4/(3*\pi))\cos((3*\pi/32)*n);$
$x_5=(4/(5*\pi))\cos((5*\pi/32)*n);$
$x_7=(4/(7*\pi))\cos((7*\pi/32)*n);$
$x_9=(4/(9*\pi))\cos((9*\pi/32)*n);$
$x_{11}=(4/(11*\pi))\cos((11*\pi/32)*n);$
$x_{13}=(4/(13*\pi))\cos((13*\pi/32)*n);$
$x_{15}=(4/(15*\pi))\cos((15*\pi/32)*n);$
$x_{17}=(4/(17*\pi))\cos((17*\pi/32)*n);$
$x_{19}=(4/(19*\pi))\cos((19*\pi/32)*n);$
$x_{21}=(4/(21*\pi))\cos((21*\pi/32)*n);$
$x_{23}=(4/(23*\pi))\cos((23*\pi/32)*n);
Image Sampling and the Spatial Frequency Concept

\[ x_{25} = \frac{4}{(25\pi)} \cos\left(\frac{25\pi}{32} n\right) \]
\[ x_{27} = \frac{4}{(27\pi)} \cos\left(\frac{27\pi}{32} n\right) \]
\[ x_{29} = \frac{4}{(29\pi)} \cos\left(\frac{29\pi}{32} n\right) \]
\[ x_{31} = \frac{4}{(31\pi)} \cos\left(\frac{31\pi}{32} n\right) \]
\[ x = x_{1} - x_{3} + x_{5} - x_{7} + x_{9} - x_{11} + x_{13} - x_{15} + x_{17} - x_{19} + x_{21} - x_{23} + x_{25} - x_{27} + x_{29} - x_{31} \]
\[ \text{stem}(x_{1}); \]
\[ \text{figure,stem}(x_{3}); \]
\[ \text{figure,stem}(x_{5}); \]
\[ \text{figure,stem}(x_{7}); \]
\[ \text{figure,stem}(x_{9}); \]
\[ \text{figure,stem}(x_{11}); \]
\[ \text{figure,stem}(x_{13}); \]
\[ \text{figure,stem}(x_{15}); \]
\[ \text{figure,stem}(x); \]  \text{figure,stem}(x); \text{title('One period of 1024x1024 size image I(n)'\n}); \]
Image Sampling and the Spatial Frequency Concept

One period of 1024x1024 size image $|I(n)|$
Image Sampling and the Spatial Frequency Concept

close all
x=0:1/25000:2/62.5;
I1=(4/pi)*cos((125*pi)*x);I3=(4/(3*pi))*cos((3*125*pi)*x);
I5=(4/(5*pi))*cos((5*125*pi)*x);I7=(4/(7*pi))*cos((7*125*pi)*x);
I9=(4/(9*pi))*cos((9*125*pi)*x);I11=(4/(11*pi))*cos((11*125*pi)*x);
I13=(4/(13*pi))*cos((13*125*pi)*x);I15=(4/(15*pi))*cos((15*125*pi)*x);
I17=(4/(17*pi))*cos((17*125*pi)*x);I19=(4/(19*pi))*cos((19*125*pi)*x);
I21=(4/(21*pi))*cos((21*125*pi)*x);I23=(4/(23*pi))*cos((23*125*pi)*x);
I25=(4/(25*pi))*cos((25*125*pi)*x);I27=(4/(27*pi))*cos((27*125*pi)*x);
I29=(4/(29*pi))*cos((29*125*pi)*x);I31=(4/(31*pi))*cos((31*125*pi)*x);
I=I1-I3+I5-I7+I9-I11+I13-I15+I17-I19+I21-I23+I25-I27+I29-I31;
figure,plot(I);title('One period of I(x) drawn using 31 harmonics');
J=I
for k = 1:800
    J=[J; I];
end
figure,mesh(J)
figure,imshow(J,[-1.25 1.25]'notruesize')
Image Sampling and the Spatial Frequency Concept