Discrete Time Fourier Transform (DTFT)
Discrete Time Fourier Transform (DTFT)

- The DTFT is the Fourier transform of choice for analyzing infinite-length signals and systems.

- Useful for conceptual, pencil-and-paper work, but not Matlab friendly (infinitely-long vectors).

- Properties are very similar to the Discrete Fourier Transform (DFT) with a few caveats.

- We will derive the DTFT as the limit of the DFT as the signal length $N \to \infty$. 
Recall: DFT (Unnormalized)

- **Analysis (Forward DFT)**
  - Choose the DFT coefficients $X[k]$ such that the synthesis produces the signal $x$
  - The weight $X[k]$ measures the similarity between $x$ and the harmonic sinusoid $s_k$
  - Therefore, $X[k]$ measures the “frequency content” of $x$ at frequency $k$

  \[
  X_u[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn}
  \]

- **Synthesis (Inverse DFT)**
  - Build up the signal $x$ as a linear combination of harmonic sinusoids $s_k$ weighted by the DFT coefficients $X[k]$

  \[
  x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_u[k] e^{j \frac{2\pi}{N} kn}
  \]
The Centered DFT

- Both $x[n]$ and $X[k]$ can be interpreted as periodic with period $N$, so we will shift the intervals of interest in time and frequency to be centered around $n, k = 0$

$$\frac{-N}{2} \leq n, k \leq \frac{N}{2} - 1$$

- The modified forward and inverse DFT formulas are

$$X_u[k] = \sum_{n=-N/2}^{N/2-1} x[n] e^{-j \frac{2\pi}{N} kn}, \quad -\frac{N}{2} \leq k \leq \frac{N}{2} - 1$$

$$x[n] = \frac{1}{N} \sum_{k=-N/2}^{N/2-1} X_u[k] e^{j \frac{2\pi}{N} kn}, \quad -\frac{N}{2} \leq n \leq \frac{N}{2} - 1$$
Recall: DFT Frequencies

\[ X_u[k] = \sum_{n=-N/2}^{N/2-1} x[n] e^{-j \frac{2\pi}{N} kn}, \quad -\frac{N}{2} \leq k \leq \frac{N}{2} - 1 \]

- \(X_u[k]\) measures the similarity between the time signal \(x\) and the harmonic sinusoid \(s_k\)

- Therefore, \(X_u[k]\) measures the “frequency content” of \(x\) at frequency

\[-\pi \leq \omega_k = \frac{2\pi}{N} k < \pi\]
Take It To The Limit (1)

\[ X_u[k] = \sum_{n=-N/2}^{N/2-1} x[n] e^{-j \frac{2\pi}{N} kn}, \quad -\frac{N}{2} \leq k \leq \frac{N}{2} - 1 \]

Let the signal length \( N \) increase towards \( \infty \) and study what happens to \( X_u[k] \)

**Key fact:** No matter how large \( N \) grows, the frequencies of the DFT sinusoids remain in the interval

\[-\pi \leq \omega_k = \frac{2\pi}{N} k < \pi\]
Take It To The Limit (2)

\[ X_u[k] = \sum_{n=-N/2}^{N/2-1} x[n] e^{-j \frac{2\pi}{N} kn} \]

\( N \)

Time signal \( x[n] \)

DFT \( X[k] \)

\( N = 32 \)

\( N = 64 \)

\( N = 128 \)

\( N = 256 \)
Discrete Time Fourier Transform (Forward)

As $N \to \infty$, the forward DFT converges to a function of the continuous frequency variable $\omega$ that we will call the **forward discrete time Fourier transform** (DTFT)

\[
\sum_{n=-N/2}^{N/2-1} x[n] e^{-j2\pi kn} \to \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X(\omega), \quad -\pi \leq \omega < \pi
\]

Recall: Inner product for infinite-length signals

\[
\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x[n] y[n]^*
\]

**Analysis interpretation:** The value of the DTFT $X(\omega)$ at frequency $\omega$ measures the similarity of the infinite-length signal $x[n]$ to the infinite-length sinusoid $e^{j\omega n}$
Discrete Time Fourier Transform (Inverse)

- Inverse unnormalized DFT

\[ x[n] = \frac{2\pi}{2\pi N} \sum_{k=-N/2}^{N/2-1} X_u[k] e^{j\frac{2\pi}{N}kn} \]

- In the limit as the signal length \( N \to \infty \), the inverse DFT converges in a more subtle way:

\[ e^{j\frac{2\pi}{N}kn} \to e^{j\omega n}, \quad X_u[k] \to X(\omega), \quad \frac{2\pi}{N} \to d\omega, \quad \sum_{k=-N/2}^{N/2-1} \to \int_{-\pi}^{\pi} \]

resulting in the inverse DTFT

\[ x[n] = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}, \quad \infty < n < \infty \]

- Synthesis interpretation: Build up the signal \( x \) as an infinite linear combination of sinusoids \( e^{j\omega n} \) weighted by the DTFT \( X(\omega) \)
Summary

- Discrete-time Fourier transform (DTFT)

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad -\pi \leq \omega < \pi \]

\[ x[n] = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}, \quad \infty < n < \infty \]

- The core “basis functions” of the DTFT are the sinusoids \( e^{j\omega n} \) with arbitrary frequencies \( \omega \)

- The DTFT can be derived as the limit of the DFT as the signal length \( N \to \infty \)

- The analysis/synthesis interpretation of the DFT holds for the DTFT, as do most of its properties
Eigenanalysis of
LT1 Systems
(Infinite-Length Signals)
For infinite length signals, \( \mathbf{H} \) is an infinitely large Toeplitz matrix with entries

\[
[H]_{n,m} = h[n - m]
\]

where \( h \) is the impulse response.

**Goal:** Calculate the eigenvectors and eigenvalues of \( \mathbf{H} \)

Eigenvectors \( \nu \) are input signals that emerge at the system output unchanged (except for a scaling by the eigenvalue \( \lambda \)) and so are somehow “fundamental” to the system.
Eigenvectors of LTI Systems

**Fact:** The eigenvectors of a Toeplitz matrix (LTI system) are the complex sinusoids

\[ s_\omega[n] = e^{j\omega n} = \cos(\omega n) + j \sin(\omega n), \quad -\pi \leq \omega < \pi, \quad -\infty < n < \infty \]
Sinusoids are Eigenvectors of LTI Systems

\[ s_\omega \xrightarrow{H} \lambda_\omega s_\omega \]

Prove that harmonic sinusoids are the eigenvectors of LTI systems simply by computing the convolution with input \( s_\omega \) and applying the periodicity of the sinusoids (infinite-length):

\[
s_\omega[n] * h[n] = \sum_{m=-\infty}^{\infty} s_\omega[n-m] h[m] = \sum_{m=-\infty}^{\infty} e^{j\omega(n-m)} h[m] \\
= \sum_{m=-\infty}^{\infty} e^{j\omega n} e^{-j\omega m} h[m] = \left( \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \right) e^{j\omega n} \\
= \lambda_\omega s_\omega[n] \quad \checkmark
\]
Eigenvalues of LTI Systems

- The eigenvalue $\lambda_\omega \in \mathbb{C}$ corresponding to the sinusoid eigenvector $s_\omega$ is called the **frequency response** at frequency $\omega$ since it measures how the system “responds” to $s_k$.

$$\lambda_\omega = \sum_{n=-\infty}^{\infty} h[n] e^{-\omega n} = \langle h, s_\omega \rangle = H(\omega) \text{ (DTFT of } h)$$

- Recall properties of the inner product: $\lambda_\omega$ grows/shrinks as $h$ and $s_\omega$ become more/less similar.
Eigendecomposition and Diagonalization of an LTI System

\[ x \xrightarrow{\mathcal{H}} y \]

\[ y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[n-m] x[m] \]

- While we can't explicitly display the infinitely large matrices involved, we can use the DTFT to "diagonalize" an LTI system

- Taking the DTFTs of \( x \) and \( h \)

\[ X(\omega) = \sum_{m=-\infty}^{\infty} x[n] e^{-\omega n}, \quad H(\omega) = \sum_{m=-\infty}^{\infty} h[n] e^{-\omega n} \]

we have that

\[ Y(\omega) = X(\omega)H(\omega) \]

and then

\[ y[n] = \int_{-\pi}^{\pi} Y(\omega) e^{j\omega n} \frac{d\omega}{2\pi} \]
Summary

- Complex sinusoids are the eigenfunctions of LTI systems for infinite-length signals (Toeplitz matrices)

- Therefore, the discrete time Fourier transform (DTFT) is the natural tool for studying LTI systems for infinite-length signals

- Frequency response $H(\omega)$ equals the DTFT of the impulse response $h[n]$

- Diagonalization by eigendecomposition implies

\[ Y(\omega) = X(\omega) H(\omega) \]
Discrete Time Fourier Transform Examples
Discrete Time Fourier Transform

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad -\pi \leq \omega < \pi \]

\[ x[n] = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}, \quad \infty < n < \infty \]

- The Fourier transform of choice for analyzing infinite-length signals and systems

- Useful for conceptual, pencil-and-paper work, but not Matlab friendly (infinitely-long vectors)
Compute the DTFT of the symmetrical **unit pulse** \( p[n] = \begin{cases} 1 & -M \leq n \leq M \\ 0 & \text{otherwise} \end{cases} \)

Note: Duration \( D_x = 2M + 1 \) samples

Example for \( M = 3 \)

Forward DTFT

\[
P(\omega) = \sum_{n=-\infty}^{\infty} p[n] e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} \quad \ldots
\]
DTFT of the Unit Pulse (2)

- Apply the finite geometric series formula

\[
P(\omega) = \sum_{n=-\infty}^{\infty} p[n] e^{-j\omega n} = \sum_{n=-M}^{M} e^{-j\omega n} = \sum_{n=-M}^{M} (e^{-j\omega})^n = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}
\]

- This is an answer but it is not simplified enough to make sense, so we continue simplifying

\[
P(\omega) = \frac{e^{j\omega M} - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} = \frac{e^{-j\omega/2} \left( e^{j\omega 2M+1} - e^{-j\omega 2M+1} \right)}{e^{-j\omega/2} \left( e^{j\omega/2} - e^{-j\omega/2} \right)} = \frac{2j \sin \left( \omega \frac{2M+1}{2} \right)}{2j \sin \left( \frac{\omega}{2} \right)}
\]
**DTFT of the Unit Pulse (3)**

- Simplified DTFT of the unit pulse of duration $D_x = 2M + 1$ samples

\[
P(\omega) = \frac{\sin \left( \frac{2M+1}{2} \omega \right)}{\sin \left( \frac{\omega}{2} \right)}
\]

- This is called the **Dirichlet kernel** or “digital sinc”
  - It has a shape reminiscent of the classical $\sin x/x$ sinc function, but it is $2\pi$-periodic

- If $p[n]$ is interpreted as the impulse response of the moving average system, then $P(\omega)$ is the frequency response (eigenvalues) (low-pass filter)
DTFT of a One-Sided Exponential

- Recall the impulse response of the recursive average system:  
  \[ h[n] = \alpha^n u[n], \ |\alpha| < 1 \]

- Compute the frequency response \( H(\omega) \)

- Forward DTFT

  \[
  H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}}
  \]

- Recursive system with \( \alpha = 0.8 \) is a low-pass filter

![Graph of DTFT](image-url)
Impulse Response of the Ideal Lowpass Filter (1)

- The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm \pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

- Compute the impulse response $h[n]$ given this $H(\omega)$

- Apply the inverse DTFT

$$h[n] = \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} e^{j\omega n} \frac{d\omega}{2\pi} = \frac{e^{j\omega c n} - e^{-j\omega c n}}{jn} \bigg|_{-\omega_c}^{\omega_c} = \frac{2\omega_c}{2\pi} \sin(\omega_c n)$$
The frequency response $H(\omega)$ of the ideal low-pass filter passes low frequencies (near $\omega = 0$) but blocks high frequencies (near $\omega = \pm \pi$)

$$H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = 2\omega_c \frac{\sin(\omega_c n)}{\omega_c n}$$

The infamous “sinc” function!
Summary

- DTFT of a rectangular pulse is a Dirichlet kernel

- DTFT of a one-sided exponential is a low-frequency bump

- Inverse DTFT of the ideal lowpass filter is a sinc function

- Work some examples on your own!
Discrete Time Fourier Transform of a Sinusoid
Discrete Fourier Transform (DFT) of a Harmonic Sinusoid

- Thanks to the orthogonality of the length-$N$ harmonic sinusoids, it is easy to calculate the DFT of the harmonic sinusoid $x[n] = s_l[n] = e^{j \frac{2\pi}{N} ln} / \sqrt{N}$

\[
X[k] = \sum_{n=0}^{N-1} s_l[n] e^{-j \frac{2\pi}{N} kn} = \langle s_l, s_k \rangle = \delta[k - l]
\]

So what is the DTFT of the infinite length sinusoid $e^{j \omega_0 n}$?
DTFT of an Infinite-Length Sinusoid

The calculation for the DTFT and infinite-length signals is much more delicate than for the DFT and finite-length signals.

Calculate the value $X(\omega_0)$ for the signal $x[n] = e^{j\omega_0 n}$

$$X(\omega_0) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 n} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega_0 n} = \sum_{n=-\infty}^{\infty} 1 = \infty$$

Calculate the value $X(\omega)$ for the signal $x[n] = e^{j\omega_0 n}$ at a frequency $\omega \neq \omega_0$

$$X(\omega_0) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{-j(\omega-\omega_0)n} = ???$$
One semi-rigorous way to deal with this quandary is to use the Dirac delta “function,” which is defined in terms of the following limit process

Consider the following function $d_\epsilon(\omega)$ of the continuous variable $\omega$

Note that, for all values of the width $\epsilon$, $d_\epsilon(\omega)$ always has unit area

$$\int d_\epsilon(\omega) \, d\omega = 1$$
What happens to $d_\epsilon(\omega)$ as we let $\epsilon \to 0$?

- Clearly $d_\epsilon(\omega)$ is converging toward something that is infinitely tall and infinitely narrow but still with unit area.

The safest way to handle a function like $d_\epsilon(\omega)$ is inside an integral, like so

$$
\int X(\omega) d_\epsilon(\omega) \, d\omega
$$
Dirac Delta Function (3)

As $\epsilon \to 0$, it seems reasonable that

$$\int X(\omega)\,d\epsilon(\omega)\,d\omega \xrightarrow{\epsilon \to 0} X(0)$$

and

$$\int X(\omega)\,d\epsilon(\omega - \omega_0)\,d\omega \xrightarrow{\epsilon \to 0} X(\omega_0)$$

So we can think of $d\epsilon(\omega)$ as a kind of “sampler” that picks out values of functions from inside an integral.

We describe the results of this limiting process (as $\epsilon \to 0$) as the Dirac delta "function" $\delta(\omega)$. 
Dirac Delta Function (4)

- **Dirac delta “function”** $\delta(\omega)$

We write

$$\int X(\omega) \delta(\omega) \, d\omega = X(0)$$

and

$$\int X(\omega) \delta(\omega - \omega_0) \, d\omega = X(\omega_0)$$

- **Remarks and caveats**
  - Do not confuse the Dirac delta “function” with the nicely behaved discrete delta function $\delta[n]$
  - The Dirac has lots of “delta,” but it is not really a “function” in the normal sense (it can be made more rigorous using the theory of generalized functions)
  - Colloquially, engineers will describe the Dirac delta as “infinitely tall and infinitely narrow”
Scaled Dirac Delta Function

If we scale the area of $d_{\epsilon}(\omega)$ by $L$, then it has the following effect in the limit

$$\int X(\omega) L \delta(\omega) \, d\omega = L X(0)$$

\[ \text{Diagram: } L \, d_{\epsilon}(\omega) \quad \text{and} \quad L \, \delta(\omega) \]
Back to determining the DTFT of an infinite length sinusoid

Rather than computing the DTFT of a sinusoid using the forward DTFT, we will show that an infinite-length sinusoid is the inverse DTFT of the scaled Dirac delta function $2\pi \delta(\omega - \omega_0)$

$$\int_{-\pi}^{\pi} 2\pi \delta(\omega - \omega_0) e^{j\omega_n} \frac{d\omega}{2\pi} = e^{j\omega_0 n}$$

Thus we have the (rather bizarre) DTFT pair

$$e^{j\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \delta(\omega - \omega_0)$$
DTFT of Real-Valued Sinusoids

■ Since

\[
\cos(\omega_0 n) = \frac{1}{2} (e^{j\omega_0 n} + e^{-j\omega_0 n})
\]

we can calculate its DTFT as

\[
\cos(\omega_0 n) \xrightarrow{\text{DTFT}} \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)
\]

■ Since

\[
\sin(\omega_0 n) = \frac{1}{2j} (e^{j\omega_0 n} - e^{-j\omega_0 n})
\]

we can calculate its DTFT as

\[
\sin(\omega_0 n) \xrightarrow{\text{DTFT}} \frac{\pi}{j} \delta(\omega - \omega_0) + \frac{\pi}{j} \delta(\omega + \omega_0)
\]
The DTFT would be of limited utility if we could not compute the transform of an infinite-length sinusoid.

Hence, the Dirac delta “function” (or something else) is a necessary evil.

The Dirac delta has infinite energy (2-norm); but then again so does an infinite-length sinusoid.
Discrete Time Fourier Transform
Properties
Recall: Discrete-Time Fourier Transform (DTFT)

- **Forward DTFT (Analysis)**
  \[ X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad -\pi \leq \omega < \pi \]

- **Inverse DTFT (Synthesis)**
  \[ x[n] = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi}, \quad \infty < n < \infty \]

- **DTFT pair**
  \[ x[n] \xrightarrow{\text{DTFT}} X(\omega) \]
The DTFT is Periodic

- We defined the DTFT over an interval of $\omega$ of length $2\pi$, but it can also be interpreted as periodic with period $2\pi$
  \[ X(\omega) = X(\omega + 2\pi k), \quad k \in \mathbb{Z} \]

- Proof
  \[
  X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega+2\pi k)n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} e^{-j2\pi kn} = X(\omega) \checkmark
  \]
DTFT Frequencies

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad -\pi \leq \omega < \pi \]

- \( X(\omega) \) measures the similarity between the time signal \( x \) and a sinusoid \( e^{j\omega n} \) of frequency \( \omega \)

- Therefore, \( X(\omega) \) measures the “frequency content” of \( x \) at frequency \( \omega \)
DFT Frequencies and Periodicity

- Periodicity of DFT means we can treat frequencies mod $2\pi$

- $X(\omega)$ measures the “frequency content” of $x$ at frequency $(\omega)_{2\pi}$
**DTFT Frequency Ranges**

- Periodicity of DTFT means every length-$2\pi$ interval of $\omega$ carries the same information

- Typical interval 1: $0 \leq \omega < 2\pi$

- Typical interval 2: $-\pi \leq \omega < \pi$ (more intuitive)
DTFT and Time Shift

- If \( x[n] \) and \( X(\omega) \) are a DTFT pair then

\[
x[n - m] \overset{\text{DFT}}{\longleftrightarrow} e^{-j\omega m}X(\omega)
\]

- Proof: Use the change of variables \( r = n - m \)

\[
\sum_{n=-\infty}^{\infty} x[n - m] e^{-j\omega n} = \sum_{r=-\infty}^{\infty} x[r] e^{-j\omega (r+m)} = \sum_{r=-\infty}^{\infty} x[r] e^{-j\omega r} e^{-j\omega m} \\
= e^{-j\omega m} \sum_{r=-\infty}^{\infty} x[r] e^{-j\omega r} = e^{-j\omega m} X(\omega) \quad \checkmark
\]
DTFT and Modulation

- If \( x[n] \) and \( X(\omega) \) are a DFT pair then

\[
e^{j\omega_0 n} x[n] \overset{\text{DFT}}{\longleftrightarrow} X(\omega - \omega_0)
\]

- Remember that the DTFT is \(2\pi\)-periodic, and so we can interpret the right hand side as \(X((\omega - \omega_0)2\pi)\)

- Proof:

\[
\sum_{n=-\infty}^{\infty} e^{j\omega_0 n} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega-\omega_0)n} = X(\omega - \omega_0)
\]
DTFT and Convolution

\[ y[n] = x[n] \ast h[n] = \sum_{m=-\infty}^{\infty} h[n-m]x[m] \]

- If

\[ x[n] \xleftarrow{\text{DTFT}} X(\omega), \quad h[n] \xleftarrow{\text{DTFT}} H(\omega), \quad y[n] \xleftarrow{\text{DTFT}} Y(\omega) \]

then

\[ Y(\omega) = H(\omega) X(\omega) \]

- Convolution in the time domain = multiplication in the frequency domain
The DTFT is Linear

It is trivial to show that if

\[ x_1[n] \overset{\text{DTFT}}{\leftrightarrow} X_1(\omega) \quad x_2[n] \overset{\text{DTFT}}{\leftrightarrow} X_2(\omega) \]

then

\[ \alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{\text{DFT}}{\leftrightarrow} \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega) \]
DTFT Symmetry Properties

- The sinusoids $e^{j\omega n}$ of the DTFT have symmetry properties:
  \[
  \text{Re} \left( e^{j\omega n} \right) = \cos(\omega n) \quad \text{(even function)}
  \]
  \[
  \text{Im} \left( e^{j\omega n} \right) = \sin(\omega n) \quad \text{(odd function)}
  \]

- These induce corresponding symmetry properties on $X(\omega)$ around the frequency $\omega = 0$

- **Even** signal/DFT
  \[
  x[n] = x[-n], \quad X(\omega) = X(-\omega)
  \]

- **Odd** signal/DFT
  \[
  x[n] = -x[-n], \quad X(\omega) = -X(-\omega)
  \]

- Proofs of the symmetry properties are identical to the DFT case; omitted here
# DFT Symmetry Properties Table

| $x[n]$     | $X(\omega)$          | Re($X(\omega)$) | Im($X(\omega)$) | $|X(\omega)|$ | $\angle X(\omega)$ |
|------------|-----------------------|------------------|------------------|----------------|---------------------|
| real       | $X(-\omega) = X(\omega)^*$ | even             | odd              | even           | odd                 |
| real & even| real & even           | even             | zero             | even           |                     |
| real & odd | imaginary & odd       | zero             | odd              | even           |                     |
| imaginary  | $X(-\omega) = -X(\omega)^*$ | odd              | even             | even           | odd                 |
| imaginary & even | imaginary & even | zero             | even             | even           |                     |
| imaginary & odd | imaginary & odd      | odd              | zero             | even           |                     |
Summary

- DTFT is periodic with period $2\pi$
- Convolution in time becomes multiplication in frequency
- DTFT has useful symmetry properties