Consider a system having a parameter $K$ such as

$$\text{System} = \frac{s^2 + s + 1}{s^3 + 4s^2 + Ks + 1}$$

Here the unknown parameter $K$ affects the poles of the system. Note that the poles of the system are the values of $s$ when

$$s^3 + 4s^2 + Ks + 1 = 0$$
So we have 2 questions?

- **Design** What value of $K$ should I choose to meet my performance requirements such as settling time, rise time, etc.?

- **Effect of Variation** What is the effect of variation of $K$ on my system?

Say $K = 0$ then $s^3 + 4s^2 + 0s + 1 = 0$.

If $K = 1$ we have $s^3 + 4s^2 + 1s + 1 = 0$.

If $K = 2$ we have $s^3 + 4s^2 + 2s + 1 = 0$.
Pole Locations Affect the Closed-Loop Performance

We know that the closed-loop system poles affects the dynamics of the system very much (such as stability, performance).
Response w.r.t. pole locations, a reminder
Pole Locations Affect the Closed-Loop Performance

Regions of poles are important because control systems are designed for some requirements such as damping and time for exponential decay.
Now we understand how pole locations affect system response. Let’s consider a mass-damper-spring system with a unity mass, \( m = 1\text{kg} \). And for instance assume that \( k = 1 \). The design requirement is that the system shall have \( \zeta \geq 0.75 \). First we shall try to satisfy the design requirement by tuning only the damping constant \( b \)

\[
-u - kx - bx = m\ddot{x}
\]

using the physical values, we obtain the transfer function\[
\frac{X(s)}{U(s)} = \frac{1}{s^2 + bs + 1}
\]
We don’t know anything about sketching root locus but if you plot the roots of

\[ s^2 + bs + 1 \]

by varying \( b \) from 0 up to \( \infty \) we obtain the following root locations shown in \textit{blue}.

For \( b = 1.5 \) we obtain a damping of \( \zeta = 0.75 \).
But we know that spring constant does not stay constant in real applications and varies with respect to temperature variations. Assume that

\[ K = \begin{cases} 
0.9 & \text{hot} \\
1.1 & \text{cold} 
\end{cases} \]

\[ \zeta = 0.75 \]

\[ b = 1.5 \text{ and } k = 1 \]

\[ b = 1.5 \text{ and } k = 0.9 \text{ but at this point } \zeta > 0.715 \]

\[ b = 1.5 \text{ and } k = 1.1 \text{ but at this point } \zeta = 0.715 \]

**Punchline**: Our design is very sensitive to parameter changes. These changes and their effects can easily be seen via root locus plots.
Root Locus is useful to detect the capabilities of the controller

By using RL diagrams we can easily observe the capabilities of our fb configuration. Assume we have a system

\[ G(s) = \frac{1}{(s + 1)(s - 2)} \]
This system is open-loop unstable due to the pole at $s = 2$. If you plot the root-locus, you get

![Root-locus diagram]

We can never stabilize this system by a pure gain $K$.

At least 1 root is always in RHP, hence unstable.

**Punchline:** So we can’t solve this problem by a static controller. We need additional poles and zeros to change the locus. Then, where to put those poles and zeros?
The generic configuration for sketching root-locus is the unity feedback configuration with the variable parameter $K$ which is set as a gain.

$$R(s) \rightarrow + \rightarrow K \rightarrow G(s) \rightarrow Y(s)$$

$$R(s) \rightarrow \frac{KG(s)}{1+KG(s)} \rightarrow Y(s)$$

**Note**: $K$ is not necessarily be a controller gain but can be any parameter of the system. Then $G(s)$ will be the parameter-free portion of the system.
We shall consider the basic form

\[ 1 + KG(s) = 0 \]

to sketch root-locus. However, MATLAB handles the following feedback configuration for RL:

\[
\frac{Y(s)}{R(s)} = \frac{G(s)}{1+KG(s)} \quad \text{No change in Characteristic Polynomial}
\]
Example

Assume we have a system

\[ T(s) = \frac{s^3 + s + 1}{s^3 + 4s^2 + ks + 1} \]

where we want to find out how the stability changes as \( k \) varies from 0 to \( \infty \). So we need to consider the denominator polynomial of \( T(s) \) and sketch the root-locus as \( k \) varies. But

\[ s^3 + 4s^2 + ks + 1 = 0 \]

is not in the form of \( 1 + kG(s) = 0 \). So we need to put this polynomial into correct form. To do this:

1. Group all \( k \) terms
2. Divide both sides of the equation by non-\( k \) terms
We have

\[ s^3 + 4s^2 + 1 + ks = 0 \]

\[ \frac{s^3 + 4s^2 + 1}{s^3 + 4s^2 + 1} + \frac{ks}{s^3 + 4s^2 + 1} = 0 \]

\[ 1 + k \frac{s}{s^3 + 4s^2 + 1} = 0 \]

\[ G(s) \]
In MATLAB

\[ G = \text{tf}([1 \ 0], [1 \ 4 \ 0 \ 1]); \]

\[ \text{rlocus}(G) \]
10 rules for sketching root locus

We have the generic form

\[ 1 + KG(s) = 1 + K \frac{Q(s)}{P(s)} = 0 \]

**RULE 1:** There are \( n \) lines (loci) of root locus diagram where \( n \) is the degree of \( Q \) or \( P \), whichever is greater.

**RULE 2:** As \( K \) increases from 0 to \( \infty \), the roots move from the poles of \( G(s) \) (roots of \( P(s) \)) to the zeros of \( G(s) \) (roots of \( Q(s) \)).

Easy to see this. Note that

\[ 1 + K \frac{Q(s)}{P(s)} = 0 \iff P(s) + KQ(s) = 0 \]

Therefore when \( K = 0 \) we have \( P(s) = 0 \) and when \( K \to \infty \) \( Q(s) \) dominates the equation. To see this divide both sides of \( 1 + K \frac{Q(s)}{P(s)} = 0 \) by \( K \). Then we have \( \frac{1}{K} + \frac{Q(s)}{P(s)} = 0 \). This implies

\[ \frac{Q(s)}{P(s)} = -\frac{1}{K} \]
10 rules for sketching root locus

\[
\frac{Q(s)}{P(s)} = -\frac{1}{K}
\]

if \( K \to \infty \) we have

\[
\frac{Q(s)}{P(s)} = 0 \iff Q(s) = 0
\]

**Question:** What happens when the number of poles \( \neq \) number of zeros? If \( \text{degree} P(s) = \text{degree} Q(s) \) then branches end at zeros.

- \( \text{degree} P(s) = \text{degree} Q(s) \)
- \( \text{degree} P(s) > \text{degree} Q(s) \)
- \( \text{degree} P(s) < \text{degree} Q(s) \)

Extra line comes from \( \infty \)
RULE 3 When roots are complex, they occur in complex conjugate pairs. This means that root-locus is symmetric w.r.t real axis.
RULE 4 At no time will the same root crossover its path.

This will never occur

This might occur
Let's assume that our open loop poles and zeros are like the following. Can we sketch the locus?

![s-plane diagram with points a and b marked]
Applying the first 3 rules:

1) 3 Poles so 3 lines
2) 2 of lines will terminate at $\infty$
3) $a$ and $b$ will move as conjugate pairs
4) Each root will not crossover its path

Can we predict how these poles move?
RULE 5 The portion of the real axis to the left of an odd number of open-loop poles and zeros are part of the loci.

RULE 6 Lines leave (break out) and enter (break in) the real axis at 90°
RULE 7 If there are not enough poles or zeros to make a pair then the extra lines go to or come from infinity.
RULE 8 Lines that go to infinity follows the asymptotes

We can plot the asymptotes with the following informations:

1. The angles of asymptotes:

   \[ \phi_A = \frac{2q + 1}{n - m} \times 180^\circ \quad q = 0, 1, 2, \ldots, (n - m - 1) \]

2. The centroid of the asymptote (real axis intersection point)

   \[ \sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n - m} \]

Note that \( n - m = \# \text{poles} - \# \text{zeros} \)
Example: If 1 line goes to infinity then
\[
\phi_A = \frac{2 \times 0 + 1}{1} \times 180^\circ = 180^\circ
\]
If 2 lines go to infinity we have
\[
\phi_{A_0} = \frac{2 \cdot 0 + 1}{2} \cdot 180^\circ = 90^\circ
\]
and
\[
\phi_{A_1} = \frac{2 \cdot 1 + 1}{2} \cdot 180^\circ = 270^\circ = -90^\circ
\]
Continuing Rules

If 3 lines go to infinity

\[ A = \frac{6}{3} = 2 \]

\[ A = 60, 180, 60 \]

Centroid
\[ \sigma_A = \frac{-6}{3} = -2 \]
\[ \phi_A = 60^\circ, 180^\circ \]

Re(s) \hspace{2cm} Im(s)
Consider \( G(s) = \frac{1}{(s+2)(s+4)(s-2)(s-4)} \)

\[
\sigma_A = \frac{(-4 - 2 + 2 + 4) - 0}{4} = 0 \quad \phi_A = 45^\circ, -45^\circ, \ldots
\]
RULE 9 If there are at least two lines that go to infinity then the sum of all roots is constant.

\[ -4 + (-2) = -6 \text{ for } K=0 \]
\[ -3 + (-3) = -6 \text{ for some } K \]
RULE 10 $K$ going from 0 to negative infinity can be drawn by reversing RULE 5 and adding $180^\circ$ to the asymptote angles.
There are some other rules which didn't consider but covered in textbooks. Those rules help us to sketch better root locus diagrams. However, I think that 10 rules that I gave in this presentation are sufficient to sketch the locus. A better and finer locus can be obtained by simulation programs such as MATLAB.
Back to our Design Problem

We have seen that we can not stabilize

$$G(s) = \frac{1}{(s + 1)(s - 2)}$$

by using a unity feedback control system with a static controller gain $K$. Now we learned that we can modify our controller and can stabilize this system. How can we do that? We know that zeros attract poles therefore adding zero to the left will solve this problem.

$$G_c(s) = K(s + 3)$$
Back to our Design Problem
Back to our Design Problem

Note that $G_c(s) = K(s + 3)$ is nothing but a PD (Proportional-Derivative) type controller which is widely used in industry.

We can’t use this configuration

![Block Diagram](Diagram.png)

PD Controller

This configuration can be used to stabilize the system

![Block Diagram](Diagram2.png)
Consider a unity-feedback system shown in figure

\[ R(s) \xrightarrow{+} K \xrightarrow{G(s)} Y(s) \]

\[ R(s) \xrightarrow{ } \frac{KG(s)}{1+KG(s)} \xrightarrow{ } Y(s) \]

where \( G(s) = \frac{1}{s(s+4)} \). Sketch the root-locus:
Consider the unity-fb system shown in the previous ex with 
\[ G(s) = \frac{1}{s(s+5)(s+10)} \]. Sketch the root-locus:
Let \( G(s) = \frac{s+3}{s(s-3)} \) in the feedback loop. Draw the root locus and find the gain \( K \) corresponding to imaginary axis crossing and the corresponding poles.
Let \( G(s) = \frac{1}{s(s+5)} \). Using Root-locus, design a proportional controller \( K \), such that closed-loop system has a damping-ratio, \( \zeta = 0.9 \). Note that \( \zeta = 0.9 \) corresponds to \( \theta = 25.8^\circ \).
Computer solution leads to:

Ex 4

Root Locus

System: G
Gain: 7.74
Pole: -2.5 - 1.22i
Damping: 0.899
Overshoot (%): 0.16
Frequency (rad/s): 2.78
<table>
<thead>
<tr>
<th>G(s)</th>
<th>Root Locus</th>
<th>G(s)</th>
<th>Root Locus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \frac{K}{sT_1 + 1} )</td>
<td><img src="1" alt="Root Locus Plot" /></td>
<td>4. ( \frac{K}{s} )</td>
<td><img src="4" alt="Root Locus Plot" /></td>
</tr>
<tr>
<td>2. ( \frac{K}{(sT_1 + 1)(sT_2 + 1)} )</td>
<td><img src="2" alt="Root Locus Plot" /></td>
<td>5. ( \frac{K}{s(sT_1 + 1)} )</td>
<td><img src="5" alt="Root Locus Plot" /></td>
</tr>
<tr>
<td>3. ( \frac{K}{(sT_1 + 1)(sT_2 + 1)(sT_3 + 1)} )</td>
<td><img src="3" alt="Root Locus Plot" /></td>
<td>6. ( \frac{K}{s(sT_1 + 1)(sT_2 + 1)} )</td>
<td><img src="6" alt="Root Locus Plot" /></td>
</tr>
</tbody>
</table>
Wrap up..

<table>
<thead>
<tr>
<th>( G(s) )</th>
<th>Root Locus</th>
<th>( G(s) )</th>
<th>Root Locus</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{K(\tau s + 1)}{s(\tau_1 + 1)(\tau_2 + 1)} )</td>
<td><img src="image1" alt="Diagram" /></td>
<td>( \frac{K(\tau s + 1)}{s^2(\tau_1 + 1)} )</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>( \frac{K}{s^2} )</td>
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<td>( \frac{K}{s^3} )</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>( \frac{K}{s^3(\tau_1 + 1)} )</td>
<td><img src="image5" alt="Diagram" /></td>
<td>( \frac{K(\tau s + 1)}{s^3} )</td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
</tbody>
</table>

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Table 7.11 (continued)

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<thead>
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<th>Root Locus</th>
<th>G(s)</th>
<th>Root Locus</th>
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<tbody>
<tr>
<td>13. ( \frac{K(sT_a + 1)(sT_b + 1)}{s^3} )</td>
<td><img src="image1.png" alt="Diagram 13" /></td>
<td>15. ( \frac{K(sT_a + 1)}{s^2(sT_1 + 1)(sT_2 + 1)} )</td>
<td><img src="image2.png" alt="Diagram 15" /></td>
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<tr>
<td>14. ( \frac{K(sT_a + 1)(sT_b + 1)}{s(sT_1 + 1)(sT_2 + 1)(sT_3 + 1)(sT_4 + 1)} )</td>
<td><img src="image3.png" alt="Diagram 14" /></td>
<td></td>
<td><img src="image2.png" alt="Diagram 15" /></td>
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